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Convexity of nodes of discrete Sturm-Liouville functions

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1. Let f be a real function defined on a set of consecutive integers $\{a, a+1, \cdots, b\}$. If the points $(k, f(k))$, $a \leq k \leq b$, are joined by straight line segments to form a broken line, then this broken line gives a representation of a continuous function, henceforth denoted by $f^*(t)$, such that $f^*(k)=f(k)$ for $a \leq k \leq b$. The zeros of $f^*(t)$ are called the nodes of $f(k)$. This paper is concerned with the convexity of nodes of functions which satisfy the following second order difference equation

$$\Delta^2 x(k-1)+q(k)x(k)=0 \tag{1}$$

where $q(k)$ is a real function defined on a set of consecutive integers to be considered. Specifically, if a nontrivial solution of (1) has three consecutive nodes t_1, t_2 and t_3, we shall be interested in the relation between the two distances t_2-t_1 and t_3-t_2 when $q(k)$ decreases over a set of consecutive integers $\{a, a+1, \cdots, b\}$ such that $a \leq t_1 \leq t_3 \leq b$.

Our work is motivated by a result of Makai [1] which states that if $x(t)$ is a nontrivial solution of the differential equation

$$x''+q(t)x=0, \quad a<x<b \tag{2}$$

with three consecutive zeros t_1, t_2 and t_3 in (a, b) and if $q(t)$ is positive, continuous and decreasing in (a, b), then

$$|x(t_2-s)| \leq |x(t_2+s)| \tag{3}$$

for $0 \leq s \leq t_2-t_1$. As can easily be seen, (3) implies the well known convexity of the zeros, *i.e.*

$$t_2-t_1 \leq t_3-t_2 . \tag{4}$$

In view of the obvious similarity between equations (1) and (2), one is tempted to conjecture that the inequality (4) also holds for the nodes of a nontrivial solution of (1) if $q(k)$ decreases over $\{a, \cdots, b\}$. This, however,

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