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Notes on complete noncompact Riemannian manifolds with convex exhaustion functions

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§ 0. Introduction

Let M be a connected, complete and noncompact Riemannian manifold without boundary, and let K_σ be the sectional curvature of M determined by a plane section σ. Every geodesic on M is parametrized by arc length; Let C_p (resp. Q_p) be the tangent cut locus (resp. the tangent first conjugate locus) with respect to a point $p \in M$, and let $C(p) = \exp_p C_p$, where $\exp_p : M_p \to M$ is the exponential map. The injectivity radius function of the exponential map is a continuous function $i : M \to \boldsymbol{R} \cup \{\infty\}$ determined by $i(p) = \inf\{d(p, q) ; q \in C(p)\}$, where d is the distance function of M induced from the Riemannian metric of M. And the injectivity radius $i(M)$ of M is defined as the infimum of $i(p)$, $p \in M$.

Toponogov ([11]) and Maeda ([8], [9]) have shown the following theorem which relates the injectivity radius with the curvature of M;

THEOREM A ([8], [9], [11]) *If the sectional curvature K_σ of M satisfies $0 < K_\sigma \leqq \lambda$ for all σ, then we have $i(p) \geqq \pi/\sqrt{\lambda}$ for all p of M.*

Recently, Sharafutdinov ([10]) has extended the above result as follows;

THEOREM B ([10]) *If M is homeomorphic to a Euclidean space and if $0 \leqq K_\sigma \leqq \lambda$ for all σ, then we have $i(p) \geqq \pi/\sqrt{\lambda}$ for all p of M.*

The proof of this estimate given in [8] and [9] is based on the fact that there is a continuous filtration of compact totally convex sets $\{C_t\}_{t \geqq 0}$ if $K_\sigma > 0$ holds for all σ (see [2]).

Now if $K_\sigma \geqq 0$ holds for all σ, then every Busemann function f_γ with respect to a ray $\gamma : [0, \infty) \to M$ is convex ([1]). Moreover if $K_\sigma \geqq 0$ for all σ, then $F = \sup\{f_\gamma : \gamma(0) = p\}$ is a convex exhaustion function and $\{F^{-1}((-\infty, t])\}_{t \geqq 0}$ gives a filtration by compact totally convex sets, where sup is taken over all rays emanating from a fixed point p of M. And it has been proved by Greene and Wu ([3]) that if $K_\sigma > 0$ then the above F can be replaced by a strongly convex exhaustion function g. Namely, g satisfies; for every compact set A in M there is a $\delta > 0$ such that the second difference