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A characterization of certain weak*-closed subalgebras of $L^{\infty}(G)$

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1. Introduction

Let G be a locally compact Hausdorff group, and let $L^{\infty}(G)$ be the usual Banach algebra. Let X be a non-zero weak*-closed linear subspace of $L^{\infty}(G)$ which is (i) left and right translation invariant, (ii) self-adjoint, and (iii) an algebra. Such subspaces X were characterized by Pathak and Shapiro [5] for LCA groups G, and by Crombez and Govaerts [1] for general locally compact Hausdorff groups G (not necessarily abelian) under the assumption that X contains the constant functions. In this paper we consider the property (ii)$'$ complemented, instead of (ii), and characterize weak*-closed linear subspaces of $L^{\infty}(G)$ with the properties (i), (ii)$'$, and (iii) for LCA groups G and compact Hausdorff groups G, not necessarily abelian. Pathak-Shapiro Theorem ([5]) and our result show that if G is a LCA group, and if X is a weak*-closed translation invariant subalgebra of $L^{\infty}(G)$, then X is complemented if and only if X is self-adjoint. Also, Crombez-Govaerts Theorem ([1]) and our result show that if G is a compact Hausdorff group, not necessarily abelian, and if X is a weak*-closed left and right translation invariant subalgebra of $L^{\infty}(G)$, then X is complemented if and only if X is self-adjoint. (See Remark 3 in section 3).

Let G be a locally compact Hausdorff group and fix left Haar measure dx on G. Let $L^{\infty}(G)$ be the class of all complex-valued essentially bounded Haar-measurable functions on G, and let $L^{1}(G)$ be the class of all complex-valued Haar-integrable functions on G. $L^{\infty}(G)$ is a commutative Banach algebra under pointwise multiplication of functions as the product. As is well-known, $L^{\infty}(G)$ is the Banach space dual of $L^{1}(G)$. For $s\in G$, left and right translation of a function f on G by s are denoted by $(L_{s}f)(x)=f(sx)$ and $(R_{s}f)(x)=f(xs)$ $(x\in G)$, respectively. A linear subspace X of $L^{\infty}(G)$ is said to be left [right, left and right] translation invariant if $L_{s}f\in X$ [$R_{s}f\in X$, $L_{s}f$ and $R_{s}f\in X$] for all $s\in G$ and $f\in X$. If G is abelian, left (and hence left and right) translation invariant subspaces of $L^{\infty}(G)$ are simply said to be translation invariant. A subset X of $L^{\infty}(G)$ is said to be self-adjoint if