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On the number of irreducible characters in a finite group

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1. Introduction

Let F be an algebraically closed field of characteristic p, and G be a finite group with a Sylow p-subgroup P. Let B be a block ideal of the group algebra FG which can be regarded as an indecomposable direct summand of FG as an $F(G\times G)$-module. We denote by $k(G)$ and $l(G)$ the number of irreducible ordinary and modular characters in G, respectively (also by $k(B)$ and $l(B)$ the number of those in the block associated with B).

In [15] the author introduced the invariant $n(B)$ that is the number of indecomposable direct summands of $B_{P\times P}$. In the present paper, we show that the inequality "$l(B)\leqq n(B)$" holds, and this inequality is closely related to the well-known result that $k(G)\leqq|G:H|k(H)$ for any subgroup H of G (see [5], [7], [14]). In section 2, we shall obtain a modular version of the above well-known result that $l(G)\leqq|G:H|l(H)$ for any subgroup H of G. When $H=P$, our result $l(B)\leqq n(B)$ provides a more explicit consequence that $l(G)\leqq|P\backslash G/P|$ (the number of (P, P)-double cosets in G) which is proved in section 3. Furthermore, in section 3, we will investigate the case that the above equality holds. In this case, for example, every projective indecomposable FG-module in B has dimension $|P|$, and every irreducible FG-module in B has dimension a power of p.

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2. Let M be a right FG-module, and H be a subgroup of G. We denote by $\mathrm{rad}_H(M)$ and $\mathrm{soc}_H(M)$ the radical and the socle of M as an FH-module. Let $r_H(M)$ and $s_H(M)$ denote the number of irreducible FH-constituents of $M/\mathrm{rad}_H(M)$ and $\mathrm{soc}_H(M)$, respectively.

LEMMA 1. *Let F be an algebraically closed field of arbitrary char-*