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Separable extensions of noncommutative rings

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1. Introduction. Separable extensions of noncommutative rings were introduced in 1966 by K. Hirata and K. Sugano [4]. In [1] Hirata isolated a special class of separable extensions, now known as H-separable extensions. These have been studied extensively in a series of papers over the last fifteen years, notably by Hirata and Sugano, themselves.

A ring A is an *H-separable* extension of a subring R if $A\otimes_R A$ is isomorphic as A, A-bimodule to a direct summand of A^n, for some positive integer n. An H-separable extension is *separable*; i.e. the multiplication map $A\otimes_R A\rightarrow A$ splits. In the case of algebras over commutative rings, H-separable extensions are closely related to Azumaya algebras. In this case, A is an H-separable extension of R if A is an Azumaya algebra over a (commutative) epimorphic extension of R.

If A is a ring with subring R we denote by C the center of A and $\Delta=A^R$, the centralizer of R in A. Then A is an H-separable extension of R if and only if Δ is finitely generated and projective as C-module, and the map ϕ: $A\otimes_R A\rightarrow \mathrm{Hom}_C(\Delta, A)$ defined by $\phi(a\otimes b)(d)=adb$, for a, $b\in A$, $d\in\Delta$, is an isomorphism. There are similarly defined maps $\Delta\otimes_C A\rightarrow \mathrm{Hom}({}_RA, {}_RA)$, $A\otimes_C\Delta\rightarrow \mathrm{Hom}(A_R, A_R)$, and $\Delta\otimes_C\Delta\rightarrow \mathrm{Hom}({}_RA_R, {}_RA_R)$, all of which are isomorphisms when A is H-separable over R. (See [12].)

In Sections 3 and 4 of this paper we generalize H-separability in two directions. We call A a *strongly separable* extension of R if $A\otimes_R A\cong K\oplus L$, where $\mathrm{Hom}_{A,A}(K, A)=(0)$ and L is a direct summand of A^n, for some positive integer n. H-separability is the case where $K=(0)$. Strong separability is equivalent to separability for algebras over a commutative ring, but not in general. We show that A is strongly separable over R if and only if Δ_C is finitely generated and projective and the map ϕ defined above is a split epimorphism. The three maps above which are isomorphisms in the H-separable case are split monomorphisms when strong separability is assumed.

If σ is an automorphism of A, denote by A_σ the A, A-bimodule which as left A-module is just A but whose right A-module structure is "twisted" by σ. Then A is a *psuedo-Galois* extension of R if there is a finite set S of R-automorphisms of A such that $A\otimes_R A$ is a direct summand of $\sum_{\sigma\in S}\oplus A_\sigma{}^n$,