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Root systems and orthogonal groups of root lattices

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0. Introduction.

The theory of root systems attached to finite dimensional complex semisimple Lie algebras has been developed much deeply (cf. [1], [3]). As a natural generalization of these Lie algebras and the corresponding root systems, the notion of Lie algebras defined by (generalized) Cartan matrices has recently been introduced (cf. [4], [10]), and the structure of associated root systems has been studied (cf. [5], [12], [13], [14]).

On the other hand, in [6] the root lattice, which is corresponding to a finite, Euclidean or low rank hyperbolic Cartan matrix, and its orthogonal group are discussed. For example, it has been confirmed that in the case when a Cartan matrix is $\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$ the orthogonal group of the associated root lattice is strictly greater than the subgroup generated by its Weyl group, diagram automorphism group and $-I$ (minus identity). Indeed the group index is 2 (cf. [6]).

The starting point of this paper is the following observation:

($\sharp$) *If Δ is a root system of type C_4, and if Γ and $O(\Gamma)$ are the root lattice and its orthogonal group respectively, then the set of all elements in $O(\Gamma)$-orbit of Δ is just a root system of type F_4.*

One can easily see this by looking at the list of root systems in [1] (cf. Section 3). In this paper we shall show the following:

($\sharp\sharp$) *If Δ is a root system associated with a finite, Euclidean or hyperbolic Cartan matrix, and if Γ and $O(\Gamma)$ are the root lattice and its orthogonal group respectively, then the set of all elements in $O(\Gamma)$-orbit of Δ forms again a root system (cf. Section 2, Theorem A).*

If an original Cartan Matrix is $\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$, for example, then we get $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$ as the Cartan matrix corresponding to the new root system