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Notes on Beurling's theorem

To Professor Mitsuru Ozawa on the occasion of his 60th birthday

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For some harmonic function on a Riemann surface with Kuramochi boundary, fine limits exist on the boundary except for a set of capacity zero (Beurling type theorem, (1), (2)). The purpose of the present paper is to improve a result in (2).

Let R be an open Riemann surface and $\{R_n\}_{n=0}^{\infty}$ be an exhaustion of R. Let R^* be the Kuramochi compactification of R and Δ_1 be the set of minimal points of $\Delta=R^*-R$. For any $p\in\Delta_1$, denote by $\mathfrak{G}_p$ the family of open sets G in R such that $R-G$ is N-thin at p. Let u be a harmonic function on R. For any $p\in\Delta_1$, then N-fine cluster set $u^N(p)$ is defined by $u^N(p)=\cap\{\overline{u(G)}: G\in\mathfrak{G}_p\}$, where the closure $\overline{u(G)}$ is taken in extended real numbers. Let F be a closed set in R with piecewise analytic boundary ∂F and G be an open set in R containing F with piecewise analytic boundary. Suppose there is a Dirichlet finite function f in $G-F$ with boundary values 1 on ∂F and 0 on ∂G. Denote by $\omega(\partial F, z, G-F)$ the unique function which gives the smallest Dirichlet integral among the functions like f. Let E be a closed set in Δ. Set $E_k=\left\{z\in R: d(z, E)\leqq\frac{1}{k}\right\}$, where d is a Kuramochi metric. Let E_k' be a closed set in R with piecewise analytic boundary such that $E_{k+1}\subset E_k'\subset E_k-\partial E_k$. Then $\omega(E\cap B(F), z, G)$ is defined by $\lim_{k\to\infty}\omega(\partial(E_k'\cap F), z, G-E_k'\cap F)$. Set $\omega(E\cap B(F), z)=\omega(E\cap B(F), z, R-R_0)$, $\omega(E, z)=\omega(E\cap B(R-R_1), z)$ and $\omega(B(F), z)=\omega(\Delta\cap B(F), z)$. A Borel set A on Δ is said to be a capacity zero if $\omega(E, z)=0$ for any closed subset E of A.

Let u be a harmonic function on R. For any open set G in R, denote by $D_G(u)$ the Dirichlet integral of u on G. Let y be a real number. If there is a number $\delta>0$ such that $D_{(a<u<b)}(u)=\infty$ for any interval (a, b) in $(y-\delta, y+\delta)$, then we call y an I-point. Denote by $\mathcal{E}=\mathcal{E}(u)$ the set of I-points. Then $\mathcal{E}$ is an open subset of real numbers. For any component $e=(c, d)$ of $\mathcal{E}$, denote by e_n the closed interval $\left[c-\frac{1}{n}, d+\frac{1}{n}\right]$.

DEFINITION 1. *A harmonic function u on R is said to be almost Dirichlet finite, if* $\lim_{n\to\infty}\omega(B(u^{-1}(e_n)), z)=0$ *on R for any component e of $\mathcal{E}$.*