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An analytical proof of Kodaira's embedding theorem for Hodge manifolds

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Introduction

The main purpose of the present paper is to give a purely analytical proof of a famous theorem due to Kodaira [4] which states that every Hodge manifold X can be holomorphically embedded in a complex projective space $P^{N}(\boldsymbol{C})$.

Our proof of the theorem is based on Kohn's harmonic theory on compact strongly pseudo-convex manifolds ([2] and [3]), and has been inspired by the proof due to Boutet de Monvel [1] of the fact that every compact strongly pseudo-convex manifold M can be holomorphically embedded in a complex affine space $\boldsymbol{C}^{N}$, provided dim $M>3$. In this paper the differentiability will always mean that of class C^{∞}. Given a vector bundle E over a manifold M, $\Gamma(E)$ will denote the space of C^{∞} cross sections of E.

1. Let $\widetilde{M}$ be an $(n-1)$-dimensional (para-compact) complex manifold, and F a holomorphic line bundle over $\widetilde{M}$. Let M' be the holomorphic $\boldsymbol{C}^{*}$-bundle associated with F, and π' the projection $M'\rightarrow\widetilde{M}$.

There are an open covering $\{U_{\alpha}\}$ of $\widetilde{M}$ and for each α a holomorphic trivialization

$$\phi_{\alpha}:\ \pi'^{-1}(U_{\alpha})\ni z\longrightarrow\left(\pi'(z),\ f_{\alpha}(z)\right)\in U_{\alpha}\times\boldsymbol{C}^{*}\ .$$

We have

$$f_{\alpha}(za)=f_{\alpha}(z)\,a,\ \ z\in\pi'^{-1}(U_{\alpha}),\ \ a\in\boldsymbol{C}^{*}\ .$$

Let $\{g_{\alpha\beta}\}$ be the system of holomorphic transition functions associated with the trivializations ϕ_{α}. Then for any α and β with $U_{\alpha}\cap U_{\beta}\neq\phi$ we have

$$f_{\alpha}(z)=g_{\alpha\beta}\left(\pi'(z)\right)f_{\beta}(z),\ \ z\in\pi'^{-1}(U_{\alpha}\cap U_{\beta})\ .$$

Let us now consider a $U(1)$-reduction M of the $\boldsymbol{C}^{*}$-bundle M'. Let π denote the projection $M\rightarrow\widetilde{M}$. Then there is a unique positive function a_{α} on U_{α} such that

$$\pi^{-1}(U_{\alpha})=\left\{z\in\pi'^{-1}(U_{\alpha})\,\middle|\,|f_{\alpha}(z)|^{2}a_{\alpha}\left(\pi'(z)\right)=1\right\}\ .$$