Hokkaido Mathematical Journal Vol. 13 (1984) p. 310–338

Lifting modules, extending modules and their applications to QF-rings

By Kiyoichi OSHIRO

(Received June 21, 1983; Revised April 19, 1984)

A right R-module M is said to be an extending module if, for any submodule A of M, there exists a direct summand A^* of M such that A^* is an essential extention of A. Dually, M is said to be a lifting module provided that, for any submodule A of M, there exists a direct summand A^* of M which is a co-essential submodule of A in M, i.e., $A^*\subseteq A$ and A/A^* is small in M/A^*.

In this paper we study the following two conditions:

(♯) *Every injective R-module is a lifting module.*

(♯)$^{\#}$ *Every projective R-module is an extending module.*

A major reason why we are interested in these (♯) and (♯)$^{\#}$ comes from the fact that these conditions are closely related to the following conditions due to Harada [13]~[15]:

(*) *Every non-small R-module contains a non-zero injective submodule.*

(*)* *Every non-cosmall R-module contains a non-zero projective direct summand.*

Indeed, we show the following theorems which are main results of this paper.

THEOREM I. *The following conditions are equivalent for a given ring* R:

1) *R satisfies* (♯).

2) *R is a right artinian ring with* (*).

3) *R is a right perfect ring and the family of all injective R-modules is closed under taking small covers.*

4) *Every R-module is expressed as a direct sum of an injective module and a small module.*

THEOREM II. *The following conditions are equivalent for a given ring* R:

1) *R satisfies* (♯)$^{\#}$.

2) *R is a ring with the ACC on right annihilator ideals and satisfies* (*)*.