Hokkaido Mathematical Journal Vol. 13 (1984) p. 271-276

Spectral orders and differences

By Yūji SAKAI

(Received February 5, 1984)

1. Introduction

The purpose of this paper is to investigate the relationship between differences of functions and majorization inequalities. More specifically, we shall extend the following theorem of Lorentz-Shimogaki-Day (see [12, Proposition 1, p. 34] and [5, Proposition (6.1) (ii), p. 941]) to the case when (X, Λ, μ) is any totally σ-finite measure space:

THEOREM L-S-D. *Let (X, Λ, μ) be a finite measure space. If f, $g\in L^{1}(X)$, then $f^{*}-g^{*}<f-g$ and $|f^{*}-g^{*}|\ll|f-g|$.*

In the above theorem, $<$ and $\ll$ mean the Hardy, Littlewood and Pólya preorders (precisely defined in chapter 2).

Our Main Theorems are Theorems 1 and 2 in chapter 2. Proofs of them are easy; but they have many important applications in analytical fields. Theorems 1 and 2 extend recent results obtained by Chong [4, the left hand side inequality of (3.7), p. 148] and by Chiti [1, Theorem, p. 24], and as a corollary to Theorem 2 (Corollary 8), we can show that, in any Orlicz spaces, convergence of a sequence $\{f_{n}\}$ to f implies convergences of $\{f_{n}^{*}\}$ to f^{*}, and $\{|f_{n}|^{*}\}$ to $|f|^{*}$, where, in general, h^{*} means the decreasing rearrangement of a measurable function h.

2. Preliminaries and statements of the Main Theorems

Let (X, Λ, μ) be a measure space. Throughout the paper, we assume that $\infty\geqq a=\mu(X)>0$ and m is Lebesgue measure on $[0, a)$. Denote by $\mathfrak{M}(X)$ the set of all extended-real valued measurable functions on X, and let $L^{1}(X)$ and $L^{\infty}(X)$ stand for the set of all integrable functions and essentially bounded functions on X respectively. Any μ a.e. equal functions are identified. To each f in $\mathfrak{M}(X)$, assign its *decreasing rearrangement* f^{*} (see [13], [2], [9] and [15]): f^{*} is a uniquely determined, non-increasing and right continuous function on $[0, a)$ which is *equidistributed* with f, that is, $d_{f}(s)\equiv\mu([f>s])=m([f^{*}>s])$ for all $s\in\boldsymbol{R}=(-\infty, \infty)$. In fact, the function f^{*} is defined by $f^{*}(t)=\sup\{s: d_{f}(s)>t\}$, provided that $\sup\emptyset=-\infty$, where $\emptyset$ denotes the empty set.