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On the generalization of the theorem of Helson and Szegö

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1. Introduction and Theorem A.

$\boldsymbol{T}$ denotes the unit circle, i. e. $\boldsymbol{T}=\{\zeta;|\zeta|=1\}$ while $\boldsymbol{Z}$ denotes the set of all integers. The normalized Lebesgue measure on $\boldsymbol{T}$ is denoted by m, i. e. $dm(\zeta)=\frac{d\theta}{2\pi}$ for $\zeta=e^{i\theta}$. Let χ stand for the identity function on $\boldsymbol{T}$, i. e. $\chi(\zeta)=\zeta$. We shall use also $\chi_{k}(\zeta)=\chi(\zeta)^{k}=\zeta^{k}$ for $k\in\boldsymbol{Z}$. For $p=1,2$, $L^{p}=L^{p}(\boldsymbol{T})$ is the Banach space of measurable functions f on $\boldsymbol{T}$ whose p-th power is m-integrable. The space L^{p} is equipped with the norm $\|f\|_{p}=\{\int_{T}|f(\zeta)|^{p}dm(\zeta)\}^{1/p}$. $L^{\infty}=L^{\infty}(\boldsymbol{T})$ is the space of essentially m-bounded functions f with the norm $\|f\|_{\infty}=\text{ess sup}\ |f(\zeta)|$. $C=C(\boldsymbol{T})$ is the space of continuous functions f on $\boldsymbol{T}$ with the norm $\|f\|_{\infty}=\max_{\zeta\in T}|f(\zeta)|$.

Given a f in L^{1}, its k-th Fourier coefficient $\hat{f}(k)$ is defined by $\hat{f}(k)=\int_{T}\chi_{-k}(\zeta)f(\zeta)dm(\zeta)$ for $k\in\boldsymbol{Z}$. For $p=1,2,\infty$, the Hardy space H^{p} (resp. the disc algebra A) is the closed subspace of functions f in L^{p} (resp. C) for which $\hat{f}(k)=0$ for all $k\leq-1$. A function f in H^{1} is called outer if

$$\log\left|\int_{T}f(\zeta)dm(\zeta)\right|=\int_{T}\log|f(\zeta)|dm(\zeta).$$

A function f in H^{∞} is called inner if $|f(\zeta)|=1$ a. e. on $\boldsymbol{T}$. The subspace spanned by the functions χ_{k}, $k\in\boldsymbol{Z}$ which we call trigonometric polynomials is denoted by $\mathscr{P}$. The subspace spanned by the functions χ_{k}, $k\geq 0$ which we call analytic polynomials is denoted by $\mathscr{P}_{+}$. For a natural number n, the subspace spanned by the functions χ_{k}, $k\leq-n$ is denoted by $\mathscr{P}_{-}^{n}$. We shall use also $\mathscr{P}_{-}=\mathscr{P}_{-}^{1}$. The analytic projection P_{+} from $\mathscr{P}$ onto $\mathscr{P}_{+}$ is defined by $\mathrm{P}_{+}f=\sum_{k\geq 0}\hat{f}(k)\chi_{k}$ for $f\in\mathscr{P}$. Let $\mathrm{P}_{-}=\mathrm{I}-\mathrm{P}_{+}$ where I is the identity operator on $\mathscr{P}$. For complex valued Borel functions $\alpha(\zeta)$ and $\beta(\zeta)$, we study the linear operator $\alpha\mathrm{P}_{+}+\beta\mathrm{P}_{-}$ which includes the analytic projection P_{+} and the Hilbert transform $\mathrm{H}=-i\mathrm{P}_{+}+i\mathrm{P}_{-}$. Let μ be a finite positive regular Borel measure on $\boldsymbol{T}$ whose Lebesgue decomposition is $d\mu=W\ dm+d\mu_{s}$. For a constant $\mathrm{M}>0$, the set of the finite positive regular Borel measures ν on $\boldsymbol{T}$ which