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Global rigidity of compact classical Lie groups

Dedicated to Professor Nobuo Shimada on his 60-th birthday

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Introduction.

The subject we treat in the present paper is the problem of global rigidity of compact classical Lie groups as Riemannian manifolds imbedded isometrically in the spaces of matrices.

Let $\boldsymbol{F}$ be one of the fields $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$, where we mean by $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$ the field of real numbers, the field of complex numbers and the field of quaternions. We denote by $G(n, \boldsymbol{F})$ the compact classical Lie group $SO(n)$, $U(n)$ or $Sp(n)$ according as $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{Q}$. Let $M(n, \boldsymbol{F})$ be the space of all $n\times n$ matrices over $\boldsymbol{F}$. Then there can be defined a euclidean inner product in $M(n, \boldsymbol{F})$ invariant under left and right multiplications of matrices contained in $G(n, \boldsymbol{F})$. With this euclidean inner product $M(n, \boldsymbol{F})$ may be regarded as a real euclidean space of dimension $n^{2}\cdot\dim_{\boldsymbol{R}}\boldsymbol{F}$. Then the induced metric on the submanifold $G=G(n, \boldsymbol{F})$ in $M(n, \boldsymbol{F})$ defines a Riemannian metric on G invariant under left and right actions of G on itself. The focus of this paper is the problem of global rigidity of the inclusion map of the Riemannian manifold $G=G(n, \boldsymbol{F})$ into $M(n, \boldsymbol{F})$, which is an isometric imbedding.

Lef $\boldsymbol{f}$ be an isometric immersion of a Riemannain manifold M into the N-dimensional euclidean space $\boldsymbol{R}^{N}$. In his paper [9], N. Tanaka showed that there is a linear differential operator L associated with $\boldsymbol{f}$ whose kernel is naturally isomorphic with the space of infinitesimal isometric deformations of $\boldsymbol{f}$. He introduced the notion of elliptic isometric immersions and then established the global rigidity theorem for elliptic isometric immersions: Assume that an isometric immersion $\boldsymbol{f}$: $M\rightarrow\boldsymbol{R}^{N}$ satisfies the following conditions: i) M is compact; ii) $\boldsymbol{f}$ is elliptic; iii) $\boldsymbol{f}$ is globally infinitesimally rigid, i. e., dim $Ker\ L=\frac{1}{2}N(N+1)$. Then if two immersions $\boldsymbol{f}_{1}$ and $\boldsymbol{f}_{2}$ of M into $\boldsymbol{R}^{N}$ lie both near to $\boldsymbol{f}$ with respect to the C^{3}-topology, and if they induce the same Riemannian metric, then there exists a unique euclidean transformation a of $\boldsymbol{R}^{N}$ such that $\boldsymbol{f}_{2}=a\boldsymbol{f}_{1}$.

In the present paper we prove the following fact: Assume that G is one