On affine symmetric spaces and the automorphism groups of product manifolds

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Introduction

Let $R$ be a manifold, and let $E$ and $F$ be two differential systems on $R$, i.e., subbundles of the tangent bundle $T(R)$ of $R$. Then the pair $(E, F)$ is called a product structure on $R$, if it satisfies the following conditions:

(P. 1) $T(R) = E + F$ (direct sum),
(P. 2) Both $E$ and $F$ are completely integrable.

A manifold $R$ equipped with a product structure $(E, F)$ is called a product manifold. Let $R$ (resp.$R'$) be a product manifold, and $(E, F)$ (resp. $(E', F')$) its product structure. By an isomorphism of $R$ onto $R'$ we mean a diffeomorphism $\phi$ of $R$ onto $R'$ such that the differential $\phi_*$ of $\phi$ sends $E$ to $E'$ and $F$ to $F'$. Clearly the product $M \times N$ of two manifolds $M$ and $N$, and hence its open submanifolds $\Omega$ become naturally product manifolds in our sense.

The main purpose of the present paper is to study the automorphism groups $\text{Aut}(\Omega)$ of product manifolds $\Omega$ together with some related problems, based on the results in our previous works [8], [12] and our recent work [13]. (For several years we have worked on the geometrizations of systems of ordinary differential equations, and the results, obtained, will be published in the near future as a series of papers under the title: On pseudo-product structures and the geometrizations of systems of ordinary differential equations, which we quote by [13])

First of all we shall explain the main theorem in the present paper.

Let $\mathfrak{g}$ be a simple graded Lie algebra of the first kind, by which we mean a graded Lie algebra (over $R$), $\mathfrak{g} = \sum \mathfrak{b}_p$, satisfying the following conditions:

1) $\dim \mathfrak{g} < \infty$, and $\mathfrak{g}$ is simple, 2) $\mathfrak{b}_{-1} = \{0\}$, and $\mathfrak{b}_p = \{0\}$ if $p \leq -2$ or $p \geq 2$. (Note that $\dim \mathfrak{b}_{-1} = \dim \mathfrak{b}_1$.) If we set $\mathfrak{h} = \mathfrak{b}_0$ and $\mathfrak{m} = \mathfrak{b}_{-1} + \mathfrak{b}_1$, we see that the system $\mathfrak{g} = \mathfrak{b} + \mathfrak{m}$ (direct sum), 2) $[\mathfrak{b}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Clearly the symmetric triple $\mathfrak{g}$ is of simple and reducible type, that is, $\mathfrak{g}$ is simple, and the linear isotropy representation of $\mathfrak{b}$ on $\mathfrak{m}$ is