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SOLVABILITY OF FINITE GROUPS ADMITTING S_3 AS A FIXED-POINT-FREE GROUP OF OPERATORS

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1. Introduction

If A is a group of automorphisms of a finite group G, we say that A acts fixed-point-freely on G if $C_G(A)=1$ ($C_G(A)$ is the set of elements of G fixed by every element of A). An important theorem of Thompson states that, in this situation, if A has prime order then G is nilpotent. R. P. Martineau has shown that G must be solvable if A is any elementary abelian group. Mrs. E. W. Ralston has shown that G must be solvable if A is cyclic of order rs, r and s distinct primes. Here we prove the following result without using the Feit-Thompson theorem on the groups of odd order.

THEOREM. *Let G be a finite group admitting a fixed-point-free group of automorphisms A, where A is isomorphic to the symmetric group of degree* 3 *and* $(|G|,|A|)=1$. *Then G is solvable.*

We now discuss the proof of the theorem. We assumed that the theorem is false and take a counterexample G to the theorem of least order.

To fix ideas, set $A=<\sigma,\tau\mid\sigma^3=\tau^2=1,\tau^{-1}\sigma\tau=\sigma^{-1}>$. By Lemma 2. 1(iv), G has only one A-invariant Sylow p-subgroups of G for each prime p that divides $|G|$. Let P be the A-invariant Sylow p-subgroup of G.

In section 4, we prove that if $C_P(\sigma)=1$, then $C_G(\tau)$ has a normal p-complement. This result is important in the proof of the theorem.

In section 5, 6, 7, and 8, we prove that if P, Q be the A-invariant Sylow p-, q-subgroups, then $PQ=QP$. By P. Hall's characterization of solvable groups, G is solvable. This shows that G does not exist.

All groups considered in this paper are assumed finite. Our notation corresponds to that of Gorenstein [2]. For a prime p, we let $Syl_p(G)$ denote the set of Sylow p-subgroups of G.

2. Some preliminary results

We first quote some frequently used results.