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Operator $\Delta-aK$ on surfaces

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§ 1. Introduction

Let M be an oriented 2-dimensional complete non-compact Riemannian manifold. Let denote by $\Delta=\operatorname{trace}\nabla\nabla$ and K the laplacian and the Gauss curvature respectively. In this note, we assume that K does not vanish identically, and consider the operator $\Delta-aK$ acting on compactly supported function on M where a is a positive constant.

D. Fischer-Colbrie and R. Schoen [2] noted that the existence of a positive function f on M satisfying $\Delta f-qf=0$ is equivalent to the condition that the first eigenvalue of $\Delta-q$ be positive on each bounded domain in M where q is a function on M. This fact has many interesting applications to stable minimal immersions and some sort of surfaces of constant mean curvature.

They also showed the following fact: For every complete metric on the disc, there exists a number a_0 depending on the metric satisfying $0\leqq a_0<1$ so that for $a\leqq a_0$ there is a positive solution of $\Delta-aK$, and for $a>a_0$ there is no positive solution ([2] COROLLARY 2). They remarked that the value a_0 is $1/4$ for the Poincaré metric on the disc and that possible values of a_0 are not known for metrics of variable curvature.

Though not stated explicitly, it was proved in M. do CARMO and C. K. PENG [1] that $a_0\leqq 1/2$ for every complete metric on the disc. A. V. POGORELOV [4] proved the same result under the assumption $K\leqq 0$. He did not state this explicitly either.

We show in this note that $a_0\leqq 1/4$ for metrics of non-positive curvature.

THEOREM. *Let M be an oriented 2-dimensional complete non-compact Riemannian manifold of non-positive curvature $K\not\equiv 0$. Suppose that a is greater than $1/4$. Then there is no positive solution of $\Delta-aK$, i. e., there exists a function f with compact support which satisfies the inequality*

$$\int_M(|df|^2+aKf^2)*1<0.$$

We use the method of A. V. Pogorelov and choose a slightly different function f from that of [4].