Transitive Lie algebras admitting differential systems

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Introduction.

In this paper we define the transitive filtered Lie algebras of depth $\mu$ and prove the structure theorems on these Lie algebras.

According to Guillemin-Sternberg [2], a Lie algebra $L$ is called a transitive Lie algebra if it possesses a filtration $\{L^p\}_{p \in \mathbb{Z}}$ satisfying: 0) $L=L^{-1}$, i) $L^p \supset L^{p+1}$, ii) $[L^p, L^q] \subset L^{p+q}$, iii) $\dim L^p/L^{p+1} < \infty$, iv) $\bigcap_{p \in \mathbb{Z}} L^p = 0$, v) $L^{p+1} = \{x \in L^p | [x, L^q] \subset L^{p+q} \text{ for all } a < 0 \}$ for $p \geq 0$. As well-known, to a transitive Lie pseudo-group corresponds a transitive Lie algebra as its formal algebra and algebraic theories of transitive Lie algebras have been adequately developed by many authors, in particular by Guillemin-Sternberg [2] and Singer-Sternberg [11].

On the other hand if a transitive Lie pseudo-group acting on a manifold $M$ admits (i.e., leaves invariant) a sequence $\{D^p\}_{p<0}$ of differential systems (i.e., subbundles of the tangent bundle $TM$ of $M$) such that 0) $TM = D^{-\mu}$ for an integer $\mu \geq 1$, i) $D^p \supset D^{p+1}$, ii) $[D^p, D^q] \subset D^{p+q}$, where $D^p$ denotes the sheaf of the local sections of $D^p$, then the transitive Lie algebra $L$ corresponding to this pseudo-group admits in a natural way another filtration $\{L^p\}$ more refined than the usual one (See §1). This new filtration starts with $L^{-\mu}$ instead of $L^{-1}$:

$$L = L^{-\mu} \supset \cdots \supset L^{-1} \supset L^{0} \supset \cdots$$

and satisfies the same conditions i), ii)$\cdots$, v) as mentioned above. A Lie algebra endowed with such a filtration will be called a transitive filtered Lie algebra of depth $\mu$. If $\mu=1$ the filtration reduces to a usual one. The contact Lie algebra $C(n)$ (See §5) is a typical example of transitive Lie algebras possessing transitive filtrations of depth 2. This filtration has already played an important rôle in the classification of the infinite primitive Lie algebras ([6]). Moreover there appear many examples of transitive filtered Lie algebras of depth greater than 1 in geometry of differential systems (cf. Tanaka [12], [14]) and in higher order contact geometry (cf. Yamaguchi [15], [16]).