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Representations of chordal subalgebras of von Neumann algebras

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§ 1. Introduction

Let M be a von Neumann algebra and $\mathfrak{T}$ a σ-weakly closed subalgebra of M containing the identity of M. A σ-weakly continuous, contractive representation of $\mathfrak{T}$ is a homomorphism ρ of $\mathfrak{T}$ into the algebra of bounded linear operators $B(H)$ on a Hilbert space H such that $\rho(1)=I$ and $\|\rho(t)\|\leq \|t\|$ for all $t\in\mathfrak{T}$. Thus as an operator from $\mathfrak{T}$ to $B(H)$, $\|\rho\|=1$. In recent years the following question has attracted considerable interest: Given such a representation ρ of $\mathfrak{T}$, when is it possible to find a triple (π, V, K) where π is a (normal) $*$-representation of M on the Hilbert space K and V is an isometry mapping H into K such that

$$\rho(t)=V^{*}\pi(t)V$$

for all $t\in\mathfrak{T}$? Such a triple, should it exist, is called a *W^{*}-dilation*, or simply a *dilation*, for ρ. It was Arveson $[A]$ who found the fundamental criterion for deciding if ρ has a dilation. To state it, let $\mathfrak{T}\otimes M_{n}$ be viewed as the $n\times n$ matrices over $\mathfrak{T}$ endowed with the norm inherited from $M\otimes M_{n}$ and let ρ_{n} be the obvious extension of ρ to $\mathfrak{T}\otimes M_{n}$, mapping into $B(H)\otimes M_{n}=B(H\otimes C^{n})$. Then ρ is called *completely contractive* if and only if $\|\rho_{n}\|=1$ for all n. Arveson's dilation theorem asserts that ρ has a dilation if and only if ρ is completely contractive. In a recent paper [PPS], Paulsen, Power and Smith showed that if $\mathfrak{T}$ is a subalgebra of M_{n} that is linearly spanned by the matrix units it contains and if the support of $\mathfrak{T}$, which is the set of (i, j) such that matrix unit e_{ij} lies in $\mathfrak{T}$, satisfies a certain graph-theoretic property which they call "chordal", then every contractive representation of $\mathfrak{T}$ is completely contractive and so admits a dilation. Our objective in this note is to generalize this notion of "chordal" to the context of von Neumann algebras and to show that if $\mathfrak{T}$ is a chordal, triangular subalgebra of M in a sense to be defined in a minute, and if M is hyperfinite, then every σ-weakly contractive representation of $\mathfrak{T}$ is completely contractive.

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