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Note on separable extensions of noncommutative rings

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Introduction.

This paper is a continuation of the author's previous paper [3]. Let A be a ring and B a subring of A such that $A=B\oplus M$ as B-B-module, and assume that A is a separable extension of B. In [3] the author considered two cases of separable extensions of this type, that is, the case where $M^{2}\subset B$ and the case where $M^{2}\subset M$, and investigated the former case mainly. In this paper we will treat the latter case, and will show that, in the case where $A=B\oplus M$ such that M in an ideal of A and left B-faithful, A is a separable extension of B, if and only if M is generated by a central idempotent f of A and a separable extension of Bf (Theorem 1). In the process of the proof of this theorem we will consider the case where $A=R\oplus S$ with S a ring and R a subring of S, and the multiplication is defined by $(r, x)(s, y)=(rs, xs+ry+xy)$ for any $x, y\in S$ and $r, s\in R$. And we will show the equivalence of the following three conditions:

(a) A is a separable extension of R
(b) A is a separable extension of $R\oplus R$
(c) S is a separable extension of R (Theorem 2).

1. Throughout this paper every ring will have the identity, and all subrings of a ring will contain the identity of the ring. As for the definition and the fundamental properties of the separable extension of a noncommutative ring, see [2]. The author requires the readers to have already known them. In particular, we will use freely Propositions 2.4 and 2.5 [2]. Moreover we require the following fact: If A_{i} is a separable extension of B_{i} for $r=1,2$, then $A=A_{1}\oplus A_{2}$ is a separable extension of $B=B_{1}\oplus B_{2}$. This is obvious by $A\otimes_{B}A=A_{1}\otimes_{B_{1}}A_{1}\oplus A_{2}\otimes_{B_{2}}A_{2}$.

The following lemma has been shown in [3] and [4].

LEMMA 1. *Let A be a ring and B a subring of A such that $A=B\oplus M$ as B-B-module with $M^{2}\subset M$. If A is a separable extension of B, then M is generated by a central idempotent of A. Consequently, M is a ring with the identity.*