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Finiteness of von Neumann algebras and non-commutative L^p-spaces

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0. Introduction

Murray and von Neumann introduced their equivalence relation among projections in a von Neumann algebra and proved that a factor is finite (i. e. every projection is finite) if and only if it has a finite trace. In [2], Cuntz and Pedersen defined another equivalence relation among all positive elements in a C^*-algebra, and proved that the algebra is finite if and only if there is a separating family of finite traces.

In this paper, we introduce an equivalence relation among the positive elements of a non-commutative L^p-space associated with an arbitrary von Neumann algebra, and we study the finiteness of non-commutative L^p-spaces with respect to it.

In §1, we recall the definition of non-commutative L^p-spaces associated with an arbitrary von Neumann algebra defined by Haagerup [4]. For non-commutative L^p-spaces $L^p(N, \tau)$ arising from a semifinite von Neumann algebra N and its trace τ, the intersection $N\cap L^p(N, \tau)$ is L^p-norm dense in $L^p(N, \tau)$. Therefore one may naturally expect some similarity of their order structures between N and $L^p(N, \tau)$ even if there are significant differences, for example, the existence of an order unit. On the other hand, for non-commutative L^p-spaces $L^p(M)$ associated with an arbitrary von Neumann algebra M, it is well-known that any non-zero element in $L^p(M)$ is not bounded and that $M\cap L^p(M)=\{0\}$. Therefore we need some care to deal with them throughout the paper. In §2, we study the monotone order completeness of $L^p(M)$. Applying the result, we show in §3 that $L^p(M)$ has the asymmetric Riesz decomposition property, and we introduce an equivalence relation among the positive elements in $L^p(M)$. In §4, using the equivalence relation introduced in §3, we define a notion of finiteness of $L^p(M)$. Considering bounded linear functionals on $L^p(M)$ which satisfy the property as traces, we show that the finiteness of $L^p(M)$ agrees with that of M for the case of $1<p<\infty$.

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