Hokkaido Mathematical Journal Vol. 19 (1990) p. 289-296

On Hardy's Inequality and Paley's Gap Theorem

Sadahiro SAEKI and Hiroshi YAMAGUCHI

Dedicated to Professor Shozo Koshi on his sixtieth birthday

(Received April 10, 1989)

Let $\boldsymbol{T}=\{z\in\boldsymbol{C}:|z|=1\}$ be the circle group, and let λ be the Lebesgue measure on $\boldsymbol{T}$ normalized so that $\lambda(\boldsymbol{T})=1$. Thus the Fourier coefficients of $f\in L^{1}(\boldsymbol{T})$ are defined by

$$\hat{f}(n)=\int_{\boldsymbol{T}}z^{-n}f(z)\,d\lambda(z)\qquad\forall n\in\boldsymbol{Z}.$$

The Hardy class $H^{1}(\boldsymbol{T})$ consists of all $f\in L^{1}(\boldsymbol{T})$ such that $\hat{f}(n)=0$ for all $n<0$. The classical inequality of Hardy states that

$$(1)\qquad \sum_{n=1}^{\infty}\frac{1}{n}|\hat{f}(n)|\leq C_{1}\|f\|_{1}\qquad\forall f\in H^{1}(\boldsymbol{T}),$$

where C_{1} is a positive constant $\leq\pi$; see, e.g., K. Hoffman [2; p. 70] or A. Zygmund [5; p. 286]. On the other hand, Paley's Gap Theorem [3] asserts that given a sequence $(n_{k})_{1}^{\infty}$ of natural numbers with $\inf\{n_{k+1}/n_{k}:k\geq 1\}>1$, there exists a finite constant C_{2} such that

$$(2)\qquad \sum_{k=1}^{\infty}|\hat{f}(n_{k})|^{2}\leq C_{2}^{2}\|f\|_{1}^{2}\qquad\forall f\in H^{1}(\boldsymbol{T}).$$

For a generalization of (2) to connected compact abelian groups, we refer to W. Rudin [4; p. 213]. In the present paper, we shall give some generalizations of these well known results both in the classical setting and the abstract setting.

Let α be a Borel measurable function on $\boldsymbol{T}$ such that $|\alpha|=1$ almost everywhere. Given $f\in L^{1}(\boldsymbol{T})$, let $\alpha^{*}f$ denote the complex measure on $\boldsymbol{T}$ defined by

$$(3)\qquad \int h\,d(\alpha^{*}f)=\int(h\circ\alpha)f\,d\lambda$$

for all bounded Borel functions h on $\boldsymbol{T}$. In other words, $\alpha^{*}f$ is the image measure of $f\lambda$ by α. Let $H_{0}^{1}(\boldsymbol{T})=\{f\in H^{1}(\boldsymbol{T}):\hat{f}(0)=0\}$. Finally recall that an inner function is an element α of $H^{1}(\boldsymbol{T})$ such that $|\alpha|=1$ almost everywhere.

THEOREM 1. *Let α,β be two functions in $H^{1}(\boldsymbol{T})$ such that $|\alpha|=1\geq|\beta|$ a.e. and $\hat{\alpha}(0)\hat{\beta}(0)=0$. Then*