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On strongly separable extensions

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Dedicated to Professor Kazuhiko HIRATA on his 60th birthday
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E. McMahon and A. C. Mewborn introduced a type of separable extensions in [4], which is called strongly separable extension. In this paper, we shall study some properties of strongly separable extensions corresponding to H-separable extensions. In § 1, we give some equivalent conditions (1.4) and in § 2, we give the commutor theorem for strongly separable extensions (2.5).

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1. Strongly separable extensions

Let R be a ring and M and N left R-modules. We shall denote $M \rightsquigarrow N$ if M is a direct sum of submodules S and K such that ${}_RS < \oplus_R(N\oplus\cdots\oplus N)$ and $\mathrm{Hom}({}_RK, {}_RN)=0$. It is easy to see that K coincides with the reject of N in M (cf. [1]), which is defined by

$$\mathrm{Rej}_M(N)=\cap\{\ker f\,;\ f\in\mathrm{Hom}({}_RM, {}_RN)\}.$$

Using this notation, we can state that a ring Λ is a strongly separable extension of a subring Γ if and only if $\Lambda\otimes_\Gamma\Lambda \rightsquigarrow \Lambda$ as Λ-Λ-medules.

LEMMA 1. 1. *Let R be a ring and M and N left R-modules such that $M \rightsquigarrow N$. Then for every R-direct summand L_1 of M, $L_1 \rightsquigarrow N$.*

PROOF. We can writ $M=L_1\oplus L_2$ and $M=S\oplus K$ with
${}_RS<\oplus_R(N\oplus\cdots\oplus N)$, $\mathrm{Hom}({}_RK, {}_RN)=0$.

Let π_1 and π_2 be projections of M to L_1 and L_2, respectively, and p_K the projection M to K. By (8.18) in [1], we have $K=\pi_1(K)\oplus\pi_2(K)$. Then the restriction of $\pi_i p_K$ to L_i is the projection of L_i to $\pi_i(K)$ ($i=1,2$). Hence we can write $L_1=S_1\oplus\pi_1(K)$ and $L_2=S_2\oplus\pi_2(K)$. Then we have $M=S\oplus K=S_1\oplus S_2\oplus K$ and $S\simeq M/K\simeq S_1\oplus S_2$. Hence $S_1<\oplus S<\oplus(N\oplus\cdots N)$. Since $\pi_1(K)<\oplus K$, $\mathrm{Hom}({}_R\pi_1(K), {}_RN)=0$. Then $L_1 \rightsquigarrow N$.

Let $\Gamma\subset B\subset\Lambda$ be rings. In case the map $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ such that $b\otimes\lambda\longmapsto b\lambda$ for $b\in B$ and $\lambda\in\Lambda$ splits as a B-Λ-map, we shall call briefly that $B\otimes_\Gamma\Lambda\longrightarrow\Lambda$ splits. In this case, by tensoring on the left with Λ over B, $\Lambda\otimes_B\Lambda<\oplus\Lambda\otimes_\Gamma\Lambda$ as Λ-Λ-modules. So, from the above lemma, we