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On the Gauss-Codazzi equations

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Introduction.

Let (M, g) be an n-dimensional Riemannian manifold. Then it is an interesting and fundamental problem to find the minimum integer m such that (M, g) can be (locally) isometrically immersed into the euclidean space $\boldsymbol{R}^{m}$. Except the case where (M, g) is a space of constant curvature, a few facts are known about the above integer m.

Related to this problem the Gauss equation brings to us a useful information. Let $\boldsymbol{f}$ be an isometric immersion of (M, g) into the euclidean space $\boldsymbol{R}^{m}$. Then the second fundamental form of $\boldsymbol{f}$ satisfies the Gauss equation, that is a purely algebraic equation essentially determined by the Riemannian curvature of (M, g) and the codimension $m-n$. In this sense, the Gauss equation may be considered as an obstruction to the existence of isometric immersions. By showing the non-existence of the solutions of the Gauss equation, many authors obtained estimates on the lower bounds of m (see [23], [19], [3] etc.).

In this paper we consider the following problem: Does the existence of solutions of the Gauss equation imply the existence of isometric immersions? As the examples that will be given in this paper show, the above problem is not true in general. There are many higher order obstructions to the existence of isometric immersions. The main purpose of this paper is to formulate two higher order obstructions called the first and second Gauss-Codazzi equations and to show the usefulness of these obstructions.

Let m be an integer with $m\geqq n$. By definition, a differentiable map $\boldsymbol{f}$ of M into $\boldsymbol{R}^{m}$ is called an *isometric immersion* if the induced metric via $\boldsymbol{f}$ coincides with the Riemannian metric g. In other words, an isometric immersion is regarded as a solution of a system of first order partial differential equations with respect to a differentiable map $\boldsymbol{f}$ of M into $\boldsymbol{R}^{m}$. We consider this system from the view point of the theory of partial differential equations. Let k be a non-negative integer and $J^{k}(\boldsymbol{R}^{m})$ be the k-th order jet bundle of local differentiable mappings of M into $\boldsymbol{R}^{m}$. The system of isometric immersions stated above defines a subvaritey P in $J^{1}(\boldsymbol{R}^{m})$. P always forms a submanifold of $J^{1}(\boldsymbol{R}^{m})$ but does not bring any