Hokkaido Mathematical Journal Vol. 19 (1990) p. 151-163

Radon, Baire, and Borel measures on compact spaces. II

H. H. SCHAEFER

(Received October 18, 1988)

Introduction

This paper is a direct sequel to [8]. Its prime purpose is to discuss, as transparently as possible and from the functional analytic angle, the various types of measures—their mutual relation and basic properties—that arise in the theory of integration on compact spaces. This is done through a novel approach which the author believes will help to better understand the cross-relations between the set theoretic and functional analytic aspects of this area. The approach is based on imbedding the space $B(X)$ of (real, bounded) Borel functions on X into the second dual $\bar{C}(X)$ of the space $C(X)$ of real continuous functions, without the *a priori* use of the Riesz representation theorem. This was done in [8, B. 2. 6]; in the interest of independent readability of this second part, the notation and main result of [8] are recalled below (C. 1). It is certainly possible to extend the present method to locally compact spaces X, but assuming X to be compact (and restricting attention to bounded Baire and Borel functions) facilitates the discussion without simplifying it beyond the acceptable.

The main result of Section C below is a topological characterization of regular Borel measures on X (C. 2. 3), which yields the Riesz representation theorem as well as several other distinguishing properties of regular Borel measures as fairly immediate consequences; for example, the fact that these measures form a band in the (Banach and order) dual $B(X)'$ of $B(X)$ (C. 2. 5). Section D is concerned with various aspects of regularity; for example, the question if a given Borel measure μ on X is regular, is already determined by the behavior of μ on $B_0(X)$ (D. 1. 1), where $B_0(X)$ is the first member in the transfinite chain $\{B_\alpha : \alpha < \omega_1\}$ of Borel classes whose union is $B(X)$. In this context, we feel that the use of transfinite ordinals in the construction of the spaces of Baire and Borel functions, is of greater intuitive appeal than the usual measurability condition. Perhaps unfortunately, the tool of ordinal numbers has been all but abandoned in favor of the (admittedly often convenient) maximality principle;