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Kakutani's example on product spectral measures

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Spectral operators of scalar-type, briefly scalar operators, were introduced by N. Dunford. They are natural analogues in Banach spaces of normal operators in Hilbert space and are precisely those operators which have an integral representation of the form $\int f dP$ for some spectral measure P and some P-integrable function f. The question of whether the sum and product of commuting scalar operators are again scalar operators was affirmatively answered in the Hilbert space setting by J. Wermer [10]. The answer is negative for Banach spaces; the first example was due to S. Kakutani [4]. A further example, in a "nicer" Banach space, was provided by C. A. McCarthy [7]. This example is usually considered as a modification of Katutani's example (which it is in some sense) and is usually quoted to show that "the same things can go wrong" in a nice separable, reflexive Banach space.

The fact is that these two examples actually illustrate somewhat different (though related) phenomena and are not simply two versions of the same point. The example of Kakutani is based on the interpretation that a spectral measure P is a uniformly bounded, multiplicative, projection-valued set function which is finitely additive and whose domain is an algebra of sets. The example of McCarthy is based on countably additive spectral measures whose domains are σ-algebras of sets. The point is that the spectral measures exhibited by Kakutani cannot be extended to spectral measures on the generated σ-algebras of sets. In particular (unlike McCarthy's example), the ranges of his spectral measures do not form a σ-complete Boolean algebra of projections in the sense of W. Bade [1] nor can they be imbedded in such a Boolean algebra of projections. These differences appear to get confused in the literature and, consequently, Kakutani's example is often misquoted; see [2; p. 192], [3; p. 2099], [6; p. 253], [7; p. 295], [8; p. 359] and [9; p. 657], for example. The purpose of this note is to draw explicit attention to this difference with the hope of clarifying it somewhat.

We begin by recalling Kakutani's construction. Let $S=S'$ be the