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Remarks on the formula for the curvature

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To Professor Noboru Tanaka on his sixtieth birthday

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Let $\omega^1, \ldots, \omega^n$ be a base of 1-forms on a manifold M. Then a Riemann metric, say g, on M has an expression

(1) $$g = g_{jk}\omega^j\omega^k.$$

Where $g=(g_{jk})$ is a $n \times n$ matrix valued function on M. We denote by $<X, Y>_g$ the inner product of tangent vectors X, Y with a common source. We set

(2) $$\xi = \omega(X), \quad \eta = \omega(Y)$$

where ω denotes the $\boldsymbol{R}^n$-valued 1-form $(\omega^1, \ldots, \omega^n)$. We set $<\xi, \eta>_g = g_{jk}\xi^j\eta^k$ so that

(3) $$<X, Y>_g = <\xi, \eta>_g.$$

Write

(4) $$d\omega^j = \frac{1}{2}\beta^j_{kl}\omega^k \wedge \omega^l, \quad \beta^j_{kl} + \beta^j_{lk} = 0.$$

Define a linear map $\beta : \boldsymbol{R}^n \to \mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ by

(5) $$(\beta(\xi)\eta)^j = \beta^j_{kl}\xi^k\eta^l.$$

Actually β should be regarded as a map of M into $\mathrm{Hom}(\boldsymbol{R}^n, \mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^n))$. Then the formula (4) can be rewritten as

(6) $$(d\omega)(X, Y) = \beta(\xi)\eta.$$

We wrote down in [3] the formula for the sectional curvature of g which is expressed by means of g_{jk} and β. In this paper we write down the formula for the curvature tensor. We then write down the O'Neill's formula [4] for the submersion using only g and β.

When we set

(7) $$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]},$$

the curvature tensor is given by