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The spheres in symmetric spaces

Dedicated to Professor Noboru Tanaka on his sixtieth birthday

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Introduction.

Our main purpose is to determine the totally geodesic spheres in every compact symmetric space. This includes finding of all the monomorphisms of the group $SU(2)\cong S^3$ into the compact Lie groups. The task is a part of the fundamental problem of determination of all the homomorphisms between symmetric spaces (1. 1); a smooth mapping $f: M\rightarrow N$ between symmetric spaces is a homomorphism if and only if f is totally geodesic, provided M is connected.

Historically, the one dimensional case of S^1 was done by E. Cartan himself [C]. The case of S^3 overlaps with Dynkin's monumental work [D] in the part where he determines all the three dimensional complex subalgebras of the complex simple Lie algebras. Wolf [W] studied the case of the spheres in the real, complex and quaternion Grassmann manifolds $G_n(\boldsymbol{R}^{2n})$, $G_n(\boldsymbol{C}^{2n})$ and $G_n(\boldsymbol{H}^{2n})$ under a certain condition to be explained later (4. 6), completing a work of Y. C. Wong. Helgason studied a sphere which corresponds to the highest root ([H], Chap. 7, § 11). Fomenko in [F-1], [F-2] and [F-3] discussed the homotopy and homology classes of totally geodesic spheres; Fomenko's book [F-4] (English translation) has just appeared. Finally, the case of the zero dimensional sphere or the pair of points was done in [CN] and [N-1]; in this case a homomorphism f: $\{o, \mathrm{p}\}\rightarrow N$ is characterized by the property that $f(p)$ is fixed by the point symmetry $s_{f(o)}$ at $f(o)$.

Our method is more geometric in a way, based on the theory under development (See [CN], [N-1] and [N-2]); one can determine the spheres by using a huge induction mechanism coming from interrelationship between the symmetric spaces, at least all those spheres in certain classes (See the end of Section 1). The article [NS] might serve as another introduction.

In § 1 we will explain our geometric method along with basic concepts. Careful reading of this section and the next will help understand