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# On global hypoellipticity of horizontal Laplacians on compact principal bundles

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## Introduction

A differential operator $L$ on a smooth $(=C^{\infty})$ manifold $\boldsymbol{M}$ is called hypoelliptic (cf. [4]), if the solutions $u$ in the sence of distribution of the equation $Lu=f$ are always smooth where $f$ is smooth. In his interesting paper [5], Hörmander gave a sufficient condition for a second order differential operator to be hypoelliptic.

First of all, we shall repeat his result in a slightly different version from the original one, for this gives a motivation of this paper. Let $C^{\infty}(\boldsymbol{M})$ (resp. $C_0^{\infty}(\boldsymbol{M})$) be the space of all smooth functions on $\boldsymbol{M}$ (resp. with compact support). Let $X_1, X_2, \ldots, X_k$ be finitely many smooth tangent vector fields on $\boldsymbol{M}$, and let $\mathfrak{h}$ be the Lie algebra generated by

$$\{\sum_{1\leq i\leq k} f_i X_i\ ;\ f_i\in C_0^{\infty}(\boldsymbol{M})\}$$

THEOREM (cf. [5]). *Suppose* $\mathfrak{h}$ *is infinitesimally transitive at every point* $p$ *of* $\boldsymbol{M}$. *Then, the differential operator* $L=\sum_{1\leq i\leq k} X_i^* X_i$ *is hypoelliptic where* $X_i^*$ *is the formal adjoint operator of* $X_i$ *with respect to an arbitrarily fixed smooth riemannian metric on* $\boldsymbol{M}$.

Now in this paper, we assume that manifolds are always connected without boundary and satisfy the second countability axiom.

In the above theorem, remark at first that every $Y\in\mathfrak{h}$ is a complete vector field. Since $\boldsymbol{M}$ is connected, the infinitesimal transitivity of $\mathfrak{h}$ at every point $p$ yields easily the transitivity of the group $H$ generated by

$$\{\exp Y\ ;\ Y=\sum f_i X_i \text{ with } f_i, \ldots, f_k\in C_0^{\infty}(\boldsymbol{M})\}$$

However it should be remarked that the converse in not necessarily true in the smooth case. This pathological phenomenon occurs in general if the Lie algebra $\mathfrak{h}$ has not the property that $\mathrm{Ad}(\exp Y)\mathfrak{h}=\mathfrak{h}$ for every $Y\in\mathfrak{h}$. So if it occurs, such a Lie algebra $\mathfrak{h}$ can not be the Lie algebra of any "infinite dimensional Lie group" (cf: [8]), that is, $\mathfrak{h}$ is non-enlargeable. A typical example of such Lie algebra is as follows (cf. [9]) ; Let