Hokkaido Mathematical Journal Vol. 21(1992) p. 51-78

Topologically extremal real algebraic surfaces in $P^2 \times P^1$ and $P^1 \times P^1 \times P^1$

Goo Ishikawa

Dedicated to Professor Haruo Suzuki on his 60th birthday

(Received April 12, 1990)

0, Introduction

In this paper, from a general viewpoint, we construct surfaces in $P^2 \times P^1$ and $P^1 \times P^1 \times P^1$ defined over R having topologically extremal properties. Precisely we show that for each pair of positive integers (d, r) (resp. (d, e, r)) there exists an M-surface A in $P^2 \times P^1$ (resp. $P^1 \times P^1 \times P^1$) of degree (d, r) (resp. (d, e, r)) such that the projection $A \rightarrow P^1$ has the maximal number of real critical points (Theorem 0.1 and Corollary 0.5). Also, we show the existence of M-surfaces in $(P^2 \# (-P^2)) \times P^1$, (Corollary 0.6). Furthermore, the construction of M-surfaces in P^3 by O. Ya. Viro [V1] is explained by a similar argument as that of this paper (Theorem 0.7).

Harnack [H] pointed out that the number of components in the real locus of a curve in P^2 of degree d defined over R does not exceed $1+(1/2)(d-1)(d-2)$ and, for each d, there exists a non-singular curve in P^2 of degree d defined over R, the real locus of which has exactly $1+(1/2)(d-1)(d-2)$ components.

Hilbert, in his famous 16th problem, proposed to investigate topological restrictions for hypersurfaces in P^n of fixed degree defined over R, especially for $n=2, 3$. Amount of papers are devoted to this problem (see [G1], [V2], [W]). For instance, non-singular real curves in P^2 of degree ≤ 7 and surfaces in P^3 of degree ≤ 4 are classified topologically. To establish such classification, we first find some restrictions on topological invariants. Second, for a fixed degree, we construct real hypersurfaces of given degree, invariants of which are permitted by the ristrictions. Then, such as Harnack's result, it is the first step of the study to obtain an uniform estimate on real hypersurfaces of given degree and to show the sharpness of the estimate.

On the other hand, we may regard a real algebraic function as a 1