Hokkaido Mathematical Journal Vol. 23 (1994) p. 465-474

Almost periodic solutions of functional differential equations with infinite delays in a Banach space

Shigeo KATO
(Received December 28, 1993)

§ 1. Introduction and preliminaries

Let E be a Banach space with norm $\|\cdot\|$ and let $J=\boldsymbol{R}=(-\infty, \infty)$ or $\boldsymbol{R}_{-}=(-\infty, 0]$. We shall mean by $C(J; E)$ the set of E-valued continuous functions defined on J. By $C_B(J; E)$ we denote the set of E-valued functions continuous and bounded on J with the sup-norm $\|\cdot\|_{\infty}$. For each $t\in\boldsymbol{R}$ and $u\in C_B(\boldsymbol{R}; E)$, the symbol u_t is defined by $u_t(s)=u(t+s)$ for $s\in\boldsymbol{R}_{-}$. Clearly $u_t\in C_B(R_{-}; E)$.

With these notations, we consider in this paper the following delay-differential equation

$$(D.D.E) \quad x'=F(t, x, x_t), \quad t\in\boldsymbol{R}.$$

Here $F(t, x, \phi)$ is an E-valued function defined on $\boldsymbol{R}\times E\times C_B(\boldsymbol{R}_{-}; E)$ which satisfies some conditions mentioned precisely later. By a solution of $(D.D.E)$, we mean a continuously differentiable function u defined on $\boldsymbol{R}$ such that $u'(t)=F(t, u(t), u_t)$ for all $t\in\boldsymbol{R}$. In this paper the term "continuous" means "strongly continuous".

Recently, we proved the existence and uniqueness of a solution of $(D.D.E)$ in the case of $E=\boldsymbol{R}^n$, the n-dimensional Euclidean space. Moreover, we showed that if $F(t, x, \phi)$ is almost periodic (a.p. for short) with respect to t uniformly for (x, ϕ) in closed bounded subsets of $\boldsymbol{R}^n\times C_B(\boldsymbol{R}_{-}; \boldsymbol{R}^n)$, then $(D.D.E)$ has a unique a.p. solution ([4]). These results give an affirmative answer to the open question proposed by G. Seifert [10]. The results of [4] and [10] are essentially based on a result of Medvedev [8] which guarantees the existence of a bounded solution on $\boldsymbol{R}$ of a certain class of differential equation. The result of [8], however, can be treated in the framework of our previous papers [2, 3]. The purpose of this paper is to extend these results to the case of a functional differential equation with infinite delay in a general Banach space.

We define the functional $[\ ,\ ]: E\times E\to\boldsymbol{R}$ by

$$[x, y]=\lim_{h\to+0}(\|x+hy\|-\|x\|)/h.$$