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Algorithm of the arithmetic-geometric mean and its complex limits

(Dedicated to Professor Rentaro Agemi on his sixtieth birthday)

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Abstract. The algorithm of the arithmetic-geometric mean defines a sequence converging to its complex limit. We prove that the correspondence between those sequences and their nonzero limits is one-to-one. Our method utilizes a new proof of a certain structural theorem of the arithmetic-geometric mean. In the process, our proof clarifies the relationship of the three aspects of our problem: patterns of the algorithm leading to various arithmetic-geometric means, a subgroup of $\pi_1(\mathbb{C}\setminus\{0,1\})$ and the modular group $\Gamma_2(4)$.

Key words: arithmetic-geometric mean, modular group.

1. Introduction and the main result

The study of the arithmetic-geometric mean of two complex numbers was started by Gauss. However, its various aspects were brought to light only by later mathematicians (e.g. von David [7] and Geppert [2]). Especially in Cox [1] was given a thorough exposition on a structural theorem of the complex means, as well as a good account of its historical background. In the present paper we are concerned with the same subject, but focusing our attention on a question not touched in the above works.

Let us begin with the famous algorithm leading to an arithmetic-geometric mean. With complex numbers a and b we consider

$$\text{(AG)}\qquad \begin{aligned} &a_0 = a, \qquad b_0 = b,\\ &a_n = \frac{a_{n-1}+b_{n-1}}{2}, \quad b_n = (a_{n-1}b_{n-1})^{1/2}, \quad n = 1, 2, \dots . \end{aligned}$$

A sequence $\{(a_n, b_n)\}$ $(n = 0, 1, \dots)$ is called an *agm-sequence* for (a, b), if it satisfies the above algorithm. Because of two possible choices of b_n at every step of the algorithm there are infinitely many such agm-sequences for (a, b). It is well known that for any agm-sequence $\{(a_n, b_n)\}$ both sequences

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