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On Kato's square root problem

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Abstract. We consider abstract versions,

$$H=-\sum_{i,j=1}^{n}A_{i}c_{ij}A_{j}+\sum_{i=1}^{n}(c_{i}A_{i}+A_{i}c_{i}')+c_{0},$$

of second-order partial differential operators defined by sectorial forms on a Hilbert space $\mathcal{H}$. The A_i are closed skew-symmetric operators with a common dense domain $\mathcal{H}_1$ and the c_{ij}, c_i etc. are bounded operators on $\mathcal{H}$ with the real part of the matrix $C=(c_{ij})$ strictly positive-definite.

We assume that $D(L)\subseteq\bigcap_{i,j=1}^{n}D(A_iA_j)$ where $L=-\sum_{i=1}^{n}A_i^2$ is defined as a form on $\mathcal{H}_1\times\mathcal{H}_1$. We further assume the c_{ij} are bounded operators on one of the Sobolev spaces $\mathcal{H}_\gamma=D((I+L)^{\gamma/2})$, $\gamma\in\langle 0,1\rangle$, equipped with the graph norm. Then we prove that

$$D((\lambda I+H)^{1/2})=D((\lambda I+H^{*})^{1/2})=\mathcal{H}_1 \tag{1}$$

for all large $\lambda\in\mathbf{R}$.

As a corollary we deduce that in any unitary representation of a Lie group all second-order subelliptic operators in divergence form with Hölder continuous principal coefficients satisfy (1).

Let K be a closed maximal accretive, regular accretive, sectorial operator on the Hilbert space $\mathcal{H}$ with associated regular sesquilinear form k and $\operatorname{Re}K$ the closed maximal accretive operator associated with the real part of k. Kato [Kat1], Theorem 3.1, proved that $D(K^{\delta})=D(K^{*\delta})=D((\operatorname{Re}K)^{\delta})$ for all $\delta\in[0,1/2\rangle$ but Lions [Lio] subsequently gave an example of a closed maximal accretive operator for which $D(K^{1/2})\neq D(K^{*1/2})$. Then Kato [Kat2], Theorems 1 and 2, proved that $D(K^{1/2})=D(K^{*1/2})$ if, and only if, both $D(K^{1/2})\subseteq D(k)$ and $D(K^{*1/2})\subseteq D(k)$. More generally $D(K^{1/2})\subseteq D(k)$ if and only if $D(k)\subseteq D(K^{*1/2})$ with a similar equivalence if K and K^* are interchanged. Therefore the identity of any two of the sets $D(K^{1/2})$, $D(K^{*1/2})$, $D(k)$ implies the identity of all three. Establishing that a particular operator K satisfies these last identities has become known as Kato's square root problem, or the Kato problem.

Kato's initial interest in these questions was motivated by problems of

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