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# First variation of holomorphic forms and some applications

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**Abstract.** We study various local invariants associated with a singular holomorphic foliation on a complex surface admitting a possibly singular invariant curve. We establish the relation among them and prove/reprove formulas relating the total sum of these invariants to some global invariants of the foliation and the invariant curve.

*Key words*: singular holomorphic foliations, invariant curves, indices.

For a holomorphic vector field $v$ on a complex surface leaving a non-singular curve $C$ invariant, C. Camacho and P. Sad [CS] introduced the index of $v$ relative to $C$ and proved an index formula, which says that the total sum of the indices is equal to the Chern number of the normal bundle of $C$. After the work of a number of authors, the theory has been generalized to the case of singular invariant curves in [S], and further, to the higher dimensional case in [LS]. In [S], the index formula was proved by taking desingularization of the curve and reducing to the case of non-singular invariant curves, while the proof in [LS] involves the Chern-Weil theory, the vanishing theorem and so forth. In this article, we first give a direct proof of the index theorem for a singular foliation $\mathcal{F}$ on a complex surface leaving a (possibly singular) compact curve $C$ invariant by explicitly computing the Chern class of the normal bundle of $C$ (Theorem 1.2).

We then consider "exponent forms" for holomorphic 1-forms defining the foliation $\mathcal{F}$ and define the "variation" of $\mathcal{F}$ relative to $C$ at a singular point as the residue of an exponent form along the link of the singularity in $C$. This turns out to be a localized class of the (co)normal bundle of the foliation (Theorem 2.2). We extend the notion of the "multiplicity" of a vector field $v$ along a (locally) irreducible invariant curve [CLS] to the case of possibly reducible curves so that it coincides with the "Schwartz index" [SS] of the restriction of $v$ to the curve. After establishing the relation among

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