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A note on the Poincaré polynomial of an arrangement

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Abstract. Let $V=\mathbb{K}^{\ell}$ be a vector space, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. A *hyperplane* in V is an affine subspace of dimension $\ell-1$. An *arrangement* $\mathcal{A}$ is a finite set of hyperplanes in V. Let $L=L(\mathcal{A})$ be the set of intersections of the hyperplanes of $\mathcal{A}$, partially ordered by reverse inclusion. Let μ be the Möbius function on L, and define a rank function on L by $r(X)=\ell-\dim X$. The *Poincaré polynomial* on $\mathcal{A}$ is given by

$$\pi(\mathcal{A},t)=\sum_{X\in L}\mu(X)(-t)^{r(X)}.$$

For $X\in L$, define the combinatorial sum

$$p(X)=(-1)^{r(X)}\sum_{X\leq Z}\mu(Z)r(Z).$$

Both the Poincaré polynomial and the quantity $p(X)$ have physical interpretations in certain cases (see the work of Zaslavsky and Varchenko, respectively).

In this paper, we prove an identity involving the Poincaré polynomial and $p(X)$ and show two applications which have connections to the work of Varchenko. The first is a chamber-counting result with an interpretation when $\mathbb{K}=\mathbb{R}$, the second a result related to the Euler beta function, defined by Varchenko when $\mathbb{K}=\mathbb{C}$.

Key words: arrangement, hyperplane, Poincaré polynomial.

1. Introduction

Let $\mathbb{K}$ be a field, and let V be a vector space over $\mathbb{K}$ of dimension ℓ. A *hyperplane* H in V is an affine subspace of dimension $(\ell-1)$. An *arrangement* $\mathcal{A}$ is a finite set of hyperplanes in V. When we wish to emphasize the dimension of V, we call $\mathcal{A}$ an ℓ-arrangement. When we wish to emphasize the vector space itself, we write $(\mathcal{A},V)$ to denote the arrangement.

We refer to [3] for terminology and basic results. Let $L=L(\mathcal{A})$ be the set of intersections of the hyperplanes of $\mathcal{A}$, partially ordered by reverse inclusion. We may define a rank function on the elements (*edges*) of L by $r(X)=\operatorname{codim}X=\ell-\dim X$. We may also define a meet and a join operation on $L(\mathcal{A})$ which give it the properties of a geometric poset.

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