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Abel-Tauber theorems for Hankel and Fourier transforms and a problem of Boas

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Abstract. We prove Abel-Tauber theorems for Hankel and Fourier transforms. For example, let f be a locally integrable function on $[0, \infty)$ which is eventually decreasing to zero at infinity. Let $\rho = 3, 5, 7, \ldots$ and ℓ be slowly varying at infinity. We characterize the asymptotic behavior $f(t) \sim \ell(t)t^{-\rho}$ as $t \to \infty$ in terms of the Fourier cosine transform of f. Similar results for sine and Hankel transforms are also obtained. As an application, we give an answer to a problem of R.P. Boas on Fourier series.

Key words: Abel-Tauber theorems, Hankel transforms, Fourier transforms, Fourier series, Π-variation.

1. Introduction and results

As a prototype, we use Fourier cosine transforms to explain our problem. Let f be a locally integrable, eventually decreasing function on $[0, \infty)$ which tends to zero at infinity, and let F_c be its Fourier cosine transform. Let $\rho > 0$ and ℓ be slowly varying at infinity (see below). We are concerned with Abel-Tauber theorems which characterize the asymptotic behavior $f(t) \sim \ell(t)t^{-\rho}$ as $t \to \infty$ in terms of F_c. It turns out that the values $1, 3, 5, \ldots$ of ρ are exceptional. For $\rho \neq 1, 3, 5, \ldots$, one can obtain the desired Abel-Tauber theorems using regular variation — or Karamata theory. See Bingham-Goldie-Teugels [BGT, Ch. 4], where references to earlier work by Hardy and Rogosinski, Aljančić, Bojanić and Tomić, Vuilleumier, Zygmund and others are given. However the same theorems do not hold for $\rho = 1, 3, 5, \ldots$. These exceptional values are related to the power series expansion of the kernel $\cos x$ (see Soni-Soni [SS]).

In [I1], one of the authors showed that one could use Π-variation — or de Haan theory in the terminology of [BGT] — to obtain the desired Abel-Tauber theorem for cosine transforms when $\rho = 1$. For theorems of the same type, we refer to [I1] (cosine series and integrals), [I2] (sine series and integrals), [I3] (Fourier-Stieltjes coefficients), and Bingham-Inoue [BI]

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