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Cup products on the complete relative cohomologies of finite groups and group algebras

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Abstract. There is a theory of cup products on the complete relative cohomology of Frobenius extensions in [N1], especially, of ring extensions of group algebras of finite groups. In this paper, we construct a cup product on the complete relative cohomology of finite groups in view of a generalization of one on the Tate cohomology of finite groups. And we show that there is an isomorphism between the complete relative cohomology of finite groups and one of the group algebras, and that it preserves these cup products.

Key words: cup product, complete (co)homology, relative (co)homology, Tate cohomology.

Introduction

Let G be a finite group and K a subgroup of G. In [N1], the complete relative cohomology groups $H^r(\mathbb{Z}G, \mathbb{Z}K, A)$ and a cup product $\cup$: $H^r(\mathbb{Z}G, \mathbb{Z}K, A) \otimes H^s(\mathbb{Z}G, \mathbb{Z}K, B) \to H^{r+s}(\mathbb{Z}G, \mathbb{Z}K, A \otimes_{\mathbb{Z}G} B)$ are defined for any two-sided $\mathbb{Z}G$-modules A, B and any $r, s \in \mathbb{Z}$. On the other hand, in this paper, we define a cup product $\cup : H^r(G, K, M) \otimes H^s(G, K, N) \to H^{r+s}(G, K, M \otimes N)$ for any left G-modules M, N and any $r, s \in \mathbb{Z}$, in view of a generalization of cup product on the Tate cohomology groups of finite groups. And we will consider the relationship between these cup products.

In §1, we show that the complete relative cohomology group of finite groups has a unique cup product (Theorem 1.4), which is a generalization of the ordinary cup product on the Tate cohomology $\hat{H}^r(G, M)$. In §2, we first introduce a modified cup product $\cup_\rho : H^r(G, K, {}_\eta A) \otimes H^s(G, K, {}_\eta B) \to H^{r+s}(G, K, {}_\eta(A \otimes_{\mathbb{Z}G} B))$ for any two-sided $\mathbb{Z}G$-modules A, B and any $r, s \in \mathbb{Z}$, which is induced by the above cup product and a G-pairing $\rho : {}_\eta A \otimes {}_\eta B \to {}_\eta(A \otimes_{\mathbb{Z}G} B)$, where ${}_\eta A$ denotes the G-module A defined by the conjugation action of G. Next we show that there exists an isomorphism $\Phi^r : H^r(\mathbb{Z}G, \mathbb{Z}K, A) \xrightarrow{\sim} H^r(G, K, {}_\eta A)$ for any two-sided $\mathbb{Z}G$-module A. In Theorem 2.3, the main theorem in the paper, it is shown that the

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