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On viscosity solutions of the Hamilton-Jacobi equation

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Abstract. Comparison and uniqueness results are obtained for viscosity solutions of Hamilton-Jacobi equations. The main objective is the characterization of the value function associated with a variational problem of the Bolza type. This is accomplished, in particular, in the presence of certain conditions reminiscent of the classical Tonelli conditions.

Key words: Hamilton-Jacobi equation, viscosity solution, Bolza problem, value function, uniqueness, comparison principle.

1. Introduction and preliminaries

The concept of a viscosity solution of a partial differential equation has been intensively studied since it was introduced by M.G. Crandall and P.-L. Lions in [14, 15], and the efforts have proved immensely successful. The voluminous body of results attests the significance, for a variety of problems, of this notion of a generalized solution.

We will here be concerned with the Hamilton-Jacobi equation which has been one of the main targets of the theory of viscosity solutions since its infancy (see e.g. [4, 5, 6, 9, 12, 15, 16, 17, 22, 24] and the review article [13]). Having its historical roots in the calculus of variations and closely allied fields, the Cauchy problem for the Hamilton-Jacobi equation admits in regular cases a variational solution which is termed the value function or, in the context of classical mechanics, the action function. In favorable circumstances the value function is by the current state of the uniqueness theory necessarily the sole viscosity solution. While the collection of uniqueness theorems is indisputably substantial, it is not yet complete. To the best of the author's knowledge, the vast majority of the available theorems, by now numerous, either fail to cover or do not give satisfactory information about certain natural problems in the calculus of variations (cf. [27]). There are however exceptions: in the article [4] uniqueness among locally Lipschitz continuous solutions that are bounded below is demonstrated for a rich

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