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The diameter of the solvable graph of a finite group

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Abstract. Let G be a finite group. We define the solvable graph $\Gamma_S(G)$ as follows: the vertices are the primes dividing the order of G and two vertices p, q are joined by an edge if there is a solvable subgroup of G of order divisible by pq. We will prove that the diameter of $\Gamma_S(G)$ is less than or equal to 4 for any finite group G. We use the classification of finite simple groups.

Key words: finite simple groups, prime graphs, solvable graphs.

1. Introduction

Let G be a finite group and $\pi(G)$ the set of primes dividing the order of G. We denote by $\pi(n)$ the set of primes dividing a natural number n.

We define the prime graph $\Gamma(G)$ as follows: the vertices are elements of $\pi(G)$, and two distinct vertices p, q are joined by an edge, we write $p \sim q$, if there is an element of order pq in G. Note that $p \sim q$ if and only if there is a cyclic subgroup of G of order pq.

We define the solvable graph $\Gamma_S(G)$ as follows: the vertices are the elements of $\pi(G)$, and two distinct vertices p, q are joined by an edge, we write $p \approx q$, if there is a solvable subgroup of G of order divisible by pq. The concept of solvable graphs was defined recently in Abe-Iiyori [1].

It has been studied about the connected components of $\Gamma(G)$ in Williams [8], Iiyori and Yamaki [5], Kondrat′ev [6]. Abe and Iiyori [1] proved that $\Gamma_S(G)$ is connected. The diameter of $\Gamma(G)$ has been determined by Lucido [7]. We denote the connected components of $\Gamma(G)$ by $\pi_1, \ldots, \pi_{n(\Gamma(G))}$, where $n(\Gamma(G))$ is the number of connected components of $\Gamma(G)$. If the order of G is even, we take π_1 to be the component containing 2. Let $d(p,q)$ (resp. $d_S(p,q)$) be the distance between two vertices p, q in $\Gamma(G)$ (resp. $\Gamma_S(G)$). We can define the diameter of $\Gamma_S(G)$ as follows:

$$\operatorname{diam}(\Gamma_S(G)) = \max\{d_S(p,q) \mid p, q \in \pi(G)\}.$$

The purpose of this paper is to prove:

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