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If $S\times T$ is semiperfect, is S or T perfect?

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Abstract. The product of a perfect and a semiperfect semigroup is semiperfect. Conversely, if S and T are semigroups such that $S\times T$ is semiperfect then S and T must both be semiperfect. We consider the question whether it follows that S or T is perfect. This question can be answered in the affirmative by showing that every non-perfect semiperfect semigroup admits $\mathbf{Z}$ or $\mathbf{N}_0$ as a minor. We show that the study of the latter question can be reduced to the case of subsemigroups of a rational vector space carrying the identical involution.

Key words: semigroup, positive definite, moment function.

1. Introduction

Suppose $(S,+,*)$ is an abelian semigroup with zero and involution. Such a structure will be called a $*$-*semigroup*, abbreviated 'semigroup' when confusion is unlikely. A function $\varphi: S\to\mathbf{C}$ is *positive definite* if

$$\sum_{j,k=1}^{n} c_j\overline{c_k}\varphi(s_j+s_k^*)\geq 0$$

for every choice of $n\in\mathbf{N}$, $s_1,\ldots,s_n\in S$, and $c_1,\ldots,c_n\in\mathbf{C}$. Denote by $\mathcal{P}(S)$ the set of all positive definite functions on S.

A *character* on S is a function $\sigma: S\to\mathbf{C}$ satisfying $\sigma(0)=1$, $\sigma(s^*)=\overline{\sigma(s)}$, and $\sigma(s+t)=\sigma(s)\sigma(t)$ for all $s,t\in S$. Denote by S^* the set of all characters on S. Denote by $\mathcal{A}(S^*)$ the least σ-field of subsets of S^* rendering the mapping $\sigma\mapsto\sigma(s): S^*\to\mathbf{C}$ measurable for each $s\in S$. Denote by $F_+(S^*)$ the set of all measures defined on $\mathcal{A}(S^*)$ and integrating $\sigma\mapsto\sigma(s)$ for all $s\in S$. For $\mu\in F_+(S^*)$, define $\mathcal{L}\mu: S\to\mathbf{C}$ by

$$\mathcal{L}\mu(s)=\int_{S^*}\sigma(s)\,d\mu(\sigma)$$

for $s\in S$. A function $\varphi: S\to\mathbf{C}$ is a *moment function* if $\varphi=\mathcal{L}\mu$ for some $\mu\in F_+(S^*)$, and a moment function φ is *determinate* if there is only one

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