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# Parabolic geometries and canonical Cartan connections

Andreas ČAP and Hermann SCHICHL

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**Abstract.** Let $G$ be a (real or complex) semisimple Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a so called $|k|$-grading, i.e. a grading of the form $\mathfrak{g}=\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{k}$, such that no simple factor of $G$ is of type $A_1$. Let $P$ be the subgroup corresponding to the subalgebra $\mathfrak{p}=\mathfrak{g}_0\oplus\cdots\oplus\mathfrak{g}_k$. The aim of this paper is to clarify the geometrical meaning of Cartan connections corresponding to the pair $(G, P)$ and to study basic properties of these geometric structures.

Let $G_0$ be the (reductive) subgroup of $P$ corresponding to the subalgebra $\mathfrak{g}_0$. A principal $P$-bundle $E$ over a smooth manifold $M$ endowed with a (suitably normalized) Cartan connection $\omega\in\Omega^1(E,\mathfrak{g})$ automatically gives rise to a filtration of the tangent bundle $TM$ of $M$ and to a reduction to the structure group $G_0$ of the associated graded vector bundle to the filtered vector bundle $TM$. We prove that in almost all cases the principal $P$ bundle together with the Cartan connection is already uniquely determined by this underlying structure (which can be easily understood geometrically), while in the remaining cases one has to make an additional choice (which again can be easily interpreted geometrically) to determine the bundle and the Cartan connection.

*Key words*: parabolic geometry, Cartan connection, partially integrable almost-CR-structure, G-structure, filtered manifold.

## 1. Introduction

It is an idea that goes back to E. Cartan (see [10]) to view manifolds endowed with certain geometric structures as "curved analogs" of homogeneous spaces. More precisely, given a Lie group $G$ and a closed subgroup $H\leq G$, a generalized space corresponding to the homogeneous space $G/H$ (which is simply called a space by Cartan) is a smooth manifold $M$ of the same dimension as $G/H$, together with a principal $H$-bundle $E\to M$ over $M$, which is endowed with a Cartan connection $\omega\in\Omega^1(E,\mathfrak{g})$, that is a trivialization of the tangent bundle of $E$ which is $H$-equivariant and reproduces the generators of fundamental vector fields. For example, for Riemannian structures (which are not among the structures considered in this paper), the group $G$ is the group of motions of $\mathbb{R}^n$, $H$ is the orthogonal group $O(n)$

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