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A remark on a theorem of Y. Kurata

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Abstract. In [K] Y. Kurata proved that the Goldie torsion theory splits centrally for dual rings. Here we extend his result to semilocal rings with left essential socle such that $\mathrm{Soc}({}_RR)^2 \subseteq \mathrm{Soc}(R_R)$. An example will demonstrate that our observation extends Kurata's result.

Key words: Goldie torsion theory, central splitting, semilocal rings, essential socle.

All rings are associative rings with unit, all left (or right) R-modules are unital and all torsion theories are considered to be hereditary. The singular submodule of a left R-module M is denoted by $Z({}_RM)$. We abbreviate $S := \mathrm{Soc}({}_RR)$ and $J := \mathrm{Jac}(R)$ for the left socle resp. the Jacobson radical. We denote the left Goldie torsion theory, that is the torsion theory whose torsion free modules are exactly the nonsingular left R-modules, by τ_G (see [G, 1.14] or [AD]) and we denote the torsion submodule of a module M by $\tau_G(M)$. A torsion theory τ is called *jansian* (or TTF) if the class of τ-torsion modules is closed under taking products. Moreover a jansian torsion theory τ is called *centrally splitting* if $\tau(R)$ is a direct summand of R and τ-torsion free modules are closed under homomorphic images. (see [Be, Theorem 1]). A classical result of Alin and Dickson [AD, Theorem 3.1] states that τ_G is centrally splitting for a ring R if and only if R is a direct product of a semisimple ring and a ring with essential left singular ideal. (Alin and Dickson use the term *global dimension zero* instead of *centrally splitting*, meaning that all torsionfree modules are injective. We have that τ_G is centrally splitting if and only if all nonsingular left R-modules are injective. The sufficiency is clear (see also [G, 5.10]). The necessity follows since if nonsingular modules are closed under homomorphic images and τ_G-torsion submodules split off, then each nonsingular module must equal its injective hull. By the remark on page 201 in [AD] τ_G is also jansian.)

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