Hokkaido Mathematical Journal Vol. 30 (2001) p. 605–613

On the group-homological description of the second Johnson homomorphism

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(Received April 28, 2000)

Abstract. The Johnson homomorphisms τ_k ($k \geq 1$) give abelian quotients of a series of certain subgroups of the mapping class group of a surface. Morita constructed the refinement $\tilde{\tau}_k$ of τ_k in terms of group homology. In this paper, we describe $\tilde{\tau}_2$ explicitly and show that the reduction of $\tilde{\tau}_2$ to τ_2 does not lose any informations.

Key words: mapping class group; Johnson homomorphism; group homology.

1. Introduction

Let $\mathcal{M}_{g,1}$ be the mapping class group of a compact oriented surface $\Sigma_{g,1}$ of genus $g \geq 2$ with one boundary component. To investigate the structure of the Torelli group $\mathcal{I}_{g,1}$, which is the kernel of the classical representation

$$\mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}(2g; \mathbb{Z}),$$

Johnson defined a surjective homomorphism

$$\tau_1 : \mathcal{I}_{g,1} \longrightarrow \Lambda^3 H_1(\Sigma_{g,1}; \mathbb{Z})$$

in [2]. Moreover, he generalized it to a series of homomorphisms $\{\tau_k\}$ such that τ_{k+1} is defind on the kernel of τ_k and the target of τ_k is an abelian group denoted by $\mathcal{L}_{k+1} \otimes H$ for each k (see [3]).

As a clue to determine the image of τ_k, Morita constructed a refinement $\tilde{\tau}_k$ of the Johnson homomorphism in terms of group homology. According to his work [6], the target of $\tilde{\tau}_k$ is the third homology $H_3(N_k)$ of a nilpotent group N_k and there is an exact sequence

$$H_3(N_k) \longrightarrow \mathcal{L}_{k+1} \otimes H \longrightarrow \mathcal{L}_{k+2} \longrightarrow 0,$$

where the composition of $\tilde{\tau}_k$ with the first map is equal to τ_k. This implies that $\mathrm{Im}\,\tau_k$ is included in the kernel of the projection $\mathcal{L}_{k+1} \otimes H \rightarrow \mathcal{L}_{k+2}$. It is a natural question to ask whether the reduction of $\tilde{\tau}_k$ to τ_k lose any

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1991 Mathematics Subject Classification : Primary 57N05; Secondary 20J05, 20F34.