Hokkaido Mathematical Journal Vol. 30 (2001) p. 519–536

Analytic foliations and center problem

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(Received February 1, 1999; Revised April 26, 2000)

Abstract. We prove a real version of Lins Neto's synthesis Theorem. The technics used, allow us to give a foliation without Liouvillian first integral and which restricts to center on the fixed point set of many antiholomorphic involutions leaving $\mathcal{F}$ invariant.

Key words: holomorphic 1-forms - center - reduction of singularities - groups of diffeomorphisms - antiholomorphic involutions - Liouvillian first integral.

1. Introduction

Let $\mathcal{F}$ and σ be germs of holomorphic foliation and antiholomorphic involution at $0 \in \mathbf{C}^2$. It is well known that if $\sigma^*\mathcal{F} = \mathcal{F}$ then $\mathcal{F}$ restricts to a real foliation on the fixed point set of σ (Fix_σ). We say that $\mathcal{F}_{/\,\mathrm{Fix}_\sigma}$ is monodromic if to each germ of real analytic curve $\tau : \mathbf{R}_0^+ \to \mathrm{Fix}_{\sigma,\tau(0)}$ corresponds a Poincaré return map $\mathcal{P}$ (for t small enough the leaf of $\mathcal{F}$, which passes through $\tau(t)$ cuts again $\tau(\mathbf{R}_0^+)$ at $\mathcal{P}(\tau(t))$). When $\mathcal{P}$ is the germ of identity, we say that $\mathcal{F}_{/\,\mathrm{Fix}_\sigma}$ is a center. The simplest example of center is the one defined by the levels of the function $f(x,y) = x^2 + y^2$, or equivalently by the 1-form $\omega = x\,dx + y\,dy$. The complexification of $\mathcal{F}_\omega$, denoted $\mathcal{F}_\omega^{\mathbf{C}}$, is the germ of foliation at $0 \in \mathbf{C}^2$ defined by 1-form $\omega^{\mathbf{C}}$, whose restriction on $\mathbf{R}_0^2$ is ω. This example corresponds to the case where $\mathcal{F}_\omega^{\mathbf{C}}$ has two holomorphic invariant curves and has the following property (cf. 4.2):

1. for each antiholomorphic involution σ which does not fix any invariant curve of $\mathcal{F}_\omega^{\mathbf{C}}$ and such that $\sigma^*\mathcal{F}_\omega^{\mathbf{C}} = \mathcal{F}_\omega^{\mathbf{C}}$, $\mathcal{F}_\omega^{\mathbf{C}}{}_{/\,\mathrm{Fix}_\sigma}$ is a center.

When $\mathcal{F}_\omega^{\mathbf{C}}$ has two invariant tangent curves (node), according to Brunella [Br], the assumption of center and some generic conditions on ω ensure that there exists an elementary mutiform first integral for ω [CM]. We are interested in centers whose complexification has four invariant curves. That is the simplest case after the one described above, since the complexification of a germ of real analytic foliation which is a center has an even number of

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1991 Mathematics Subject Classification : 32S45, 32S64, 34A26.