Hokkaido Mathematical Journal Vol. 30 (2001) p. 137–161

Scattering theory and large time asymptotics of solutions to the Hartree type equations with a long range potential

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(Received August 4, 1999; Revised January 14, 2000)

Abstract. We study the scattering problem and asymptotics for large time of solutions to the Hartree type equations

$$iu_t = -\frac{1}{2}\Delta u + f(|u|^2)u,\quad (t,x)\in\mathbf{R}\times\mathbf{R}^n,\quad u(0,x)=u_0(x),\quad x\in\mathbf{R}^n,\quad n\geq 1,$$

where the nonlinear interaction term is $f(|u|^2) = V * |u|^2$, $V(x) = \lambda|x|^{-\delta}$, $\lambda\in\mathbf{R}$, $0<\delta<1$. We suppose that in the case $n\geq 2$ the initial data $u_0\in H^{n+2,0}\cap H^{0,n+2}$ and the value $\epsilon = \|u_0\|_{H^{n+2,0}} + \|u_0\|_{H^{0,n+2}}$ is sufficiently small and in one-dimensional case ($n=1$) we assume that $e^{\beta|x|}u_0\in L^2$, $\beta>0$ and the value $\epsilon = \|e^{\beta|x|}u_0\|_{L^2}$ is sufficiently small. Then we prove that there exists a unique final state $\hat{u}_+\in H^{n+2,0}$ such that the asymptotics

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\left(\frac{x}{t}\right) + O(1+t^{1-2\delta})\right) + O(t^{-n/2-\delta})$$

is true as $t\to\infty$ uniformly with respect to $x\in\mathbf{R}^n$ with the following decay estimate $\|u(t)\|_{L^p}\leq C\epsilon t^{\frac{n}{p}-\frac{n}{2}}$, for all $t\geq 1$ and for every $2\leq p\leq\infty$. Furthermore we show that for $\frac{1}{2}<\delta<1$ there exists a unique final state $\hat{u}_+\in H^{n+2,0}$ such that

$$\left\|u(t) - \exp\left(-\frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\left(\frac{x}{t}\right)\right)U(t)u_+\right\|_{L^2} = O(t^{1-2\delta})$$

for all $t\geq 1$, and the asymptotic formula

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\left(\frac{x}{t}\right)\right) + O(t^{-n/2+1-2\delta}),$$

is valid as $t\to\infty$ uniformly with respect to $x\in\mathbf{R}^n$, where $\hat{\phi}$ denotes the Fourier transform of the function ϕ, $H^{m,s} = \{\phi\in\mathcal{S}'; \|\phi\|_{m,s} = \|(1+x^2)^{s/2}(1-\Delta)^{m/2}\phi\|_{L^2} < \infty\}$, $m,s\in\mathbf{R}$. Analogous results are obtained for the following NLS equation

$$iu_t = -\frac{1}{2}\Delta u + \lambda t^{n-\delta}|u|^2 u$$

with cubic nonlinearity and growing with time coefficient, where $0<\delta<1$, $n\geq 1$.

Key words: Scattering theory, Hartree equation, long range potential.

1991 Mathematics Subject Classification : 35Q55.