Hokkaido Mathematical Journal Vol. 31 (2002) p. 219–253

A generalization of Chaitin's halting probability Ω and halting self-similar sets

Kohtaro TADAKI

(Received September 6, 2000)

Abstract. We generalize the concept of randomness in an infinite binary sequence in order to characterize the degree of randomness by a real number $D > 0$. Chaitin's halting probability Ω is generalized to Ω^D whose degree of randomness is precisely D. On the basis of this generalization, we consider the degree of randomness of each point in Euclidean space through its base-two expansion. It is then shown that the maximum value of such a degree of randomness provides the Hausdorff dimension of a self-similar set that is computable in a certain sense. The class of such self-similar sets includes familiar fractal sets such as the Cantor set, von Koch curve, and Sierpiński gasket. Knowledge of the property of Ω^D allows us to show that the self-similar subset of $[0, 1]$ defined by the halting set of a universal algorithm has a Hausdorff dimension of one.

Key words: algorithmic information theory, Kolmogorov complexity, randomness, Chaitin's Ω, Hausdorff dimension, self-similar set.

1. Introduction

The Kolmogorov complexity $H(s)$ of a finite binary sequence s is the size, in bits, of the shortest program for a universal algorithm U to calculate s. The concept of Kolmogorov complexity plays an important role in characterizing the randomness of an infinite binary sequence. In [C3], the four concepts of randomness in an infinite binary sequence (Chaitin, weak Chaitin, Martin-Löf, and Solovay randomness) are considered. These four concepts are shown to be equivalent to one another. In this paper, we first generalize these four concepts of randomness in order to deal with the degree of randomness of an infinite binary sequence. The degree of randomness is specified by a real number D with $0 < D \leq 1$. As D becomes larger, the degree of randomness increases. In the case when $D = 1$, the concept of the degree of randomness becomes the same as that of randomness. The relationship among the generalized concepts of randomness is investigated. Chaitin's halting probability Ω is an example of a random real number. We generalize Ω to Ω^D so that the degree of randomness of Ω^D is precisely D.

2000 Mathematics Subject Classification : Primary 68Q30; Secondary 28A80, 28A78.