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A generalization of the Lieb-Thirring inequalities in low dimensions

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Abstract. We give an estimate for the moments of the negative eigenvalues of elliptic operators on $\mathbb{R}^n$ in low dimensions. The estimate is a generalization of the Lieb-Thirring inequalities in one or two dimensions. We use the φ-transform decomposition of Frazier and Jawerth.

Key words: elliptic operator, eigenvalues, φ-transform, A_p-weights.

1. Introduction

For a real-valued measurable function V on $\mathbb{R}^n$ we set

$$V_+(x)=\max(V(x),0) \quad \text{and} \quad V_-(x)=\max(-V(x),0).$$

The Lieb-Thirring inequalities state

$$\sum_i |\lambda_i|^\gamma \leq c_{n,\gamma}\int_{\mathbb{R}^n} V_-^{n/2+\gamma}dx \tag{1}$$

for suitable $\gamma \geq 0$, where $\lambda_1 \leq \lambda_2 \leq \cdots$ are the negative eigenvalues of the Schrödinger operator $-\Delta+V$ on $L^2(\mathbb{R}^n)$. The inequality (1) holds if and only if

$$\begin{aligned} &\gamma \geq \frac{1}{2} && \text{for} \quad n=1,\\ &\gamma > 0 && \text{for} \quad n=2,\\ &\gamma \geq 0 && \text{for} \quad n \geq 3. \end{aligned}$$

The case $\gamma>1/2$, $n=1$, $\gamma>0$, $n\geq 2$ was proved by Lieb and Thirring ([8]). They applied the inequality (1) to the problem of the stability of matter. The case $\gamma=1/2$, $n=1$ was proved by Weidl ([18]). The case $\gamma=0$, $n\geq 3$ was established by Cwikel ([1]), Lieb ([7]) and Rozenbljum ([12], [13]). Some generalizations and variations of the Lieb-Thirring inequalities

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