*Hokkaido Mathematical Journal Vol. 33 (2004) p. 539–549*

# On the excesses of sequences of complex exponentials

Alexander KHEYFITS

(Received August 26, 2002; Revised April 24, 2003)

**Abstract.** We derive an equation for the excess in $L^2(-\pi, \pi)$ of a sequence of complex exponentials $\{e^{i\lambda_n t}\}_{-\infty}^{\infty}$ with $|\lambda_n - n| \leq \Delta < \infty$, $\forall n$, consider examples, and study the stability of the excess under small perturbations of $\lambda_n$.

*Key words*: $L^2$-completeness, excess.

## 1. Introduction

Let $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers and

$$E(\Lambda) = \{e^{i\lambda_n t}\}_{-\infty}^{\infty}.$$

The system $E(\Lambda)$ is said to be *closed* in $L^2(a, b)$ if there is no nontrivial function $f \in L^2(a, b)$ orthogonal to all functions in $E(\Lambda)$. The system $E(\Lambda)$ is said to be *complete* in $L^2(a, b)$ if each function $f \in L^2(a, b)$ can be approximated by the functions in $E(\Lambda)$ in the $L^2(a, b)$-norm.
Otherwise, the system is called *incomplete* in $L^2(a, b)$. It is known [9, Section 11] that in $L^2$ these two properties are equivalent.

The system is called *minimal* in $L^2(a, b)$ if each element of the system lies outside the closed linear span of the others. A closed minimal system is called *exact*. A closed system that remains closed if $r \geq 0$ of its terms are removed, but fails to be closed after removing $r + 1$ terms, is said to have the *excess* $r = \mathrm{exc}(E(\Lambda)) = r(\Lambda)$. If a non-closed system becomes exact after adding $-r$, $r < 0$, elements, the system is said to have the *deficiency* (negative excess) $r$. Thus, the system is exact if and only if its excess $r = 0$. If the system remains closed after removing any finite number of its terms, then the excess $r = \infty$, and if the system cannot be made closed by adding any finite number of elements, then the excess $r = -\infty$.

Since "The terminology of the subject is not uniform..." (R. Young [12, p. 16, footnote]), it should be mentioned that in terminology we follow [9].

An extensive literature devoted to the study of closed, minimal, etc.,

---

*2000 Mathematics Subject Classification* : 42C30, 30D15.