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# A Skoda-Type Integrability Theorem for Singular Monge–Ampère Measures

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Abstract. Let $\varphi$ be a plurisubharmonic function defined on an open subset of $\mathbb{C}^n$. We give a sufficient condition for the local integrability of $e^{-\varphi}$ with respect to a Monge–Ampère measure with Hölder-continuous potential $\mu = (dd^c u)^n$. This condition is expressed in terms of the Lelong numbers of $\varphi$ and the Hölder exponent of $u$.

## 1. Introduction and Main Result

Let $\Omega$ be an open subset of $\mathbb{C}^n$. Recall that a function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic* (p.s.h. for short) if $\varphi$ is upper semicontinuous, not identically $-\infty$ in some connected component of $\Omega$, and if for every complex line $L \subset \mathbb{C}^n$, the function $\varphi|_{\Omega \cap L}$ is either subharmonic in $\Omega \cap L$ or identically $-\infty$.

The *Lelong number* of $\varphi$ at $a \in \Omega$ can be defined as

$$\nu(\varphi; a) := \liminf_{z \to a, z \neq a} \frac{\varphi(z)}{\log|z-a|},$$

and it somewhat measures the singularity of $\varphi$ at $a$. This number can be characterized as $\nu(\varphi; a) = \sup\{\gamma : \varphi(z) \leq \gamma \log|z-a| + O(1) \text{ as } z \to a\}$ and is one of the most basic quantities associated with the pole of $\varphi$ at $a$. The function $z \mapsto \nu(\varphi; z)$ is upper semicontinuous with respect to the usual topology, and a deep theorem of Siu [Siu74] states that this function is also upper semicontinuous with respect to the Zariski topology, that is, for every $c > 0$, the set $\{a \in \Omega; \nu(\varphi, a) \geq c\}$ is a closed analytic subvariety of $\Omega$. For further properties and equivalent definitions of the Lelong number, the reader may consult [Dem] and [Hör07].

Another way of measuring the singularity of a p.s.h. function at a point was introduced by Demailly and Kollár [DK01] (see also [Tia87]) as a tool to study several types of algebraic and analytic objects, such as holomorphic functions, divisors, coherent ideal sheaves, positive closed currents, and so on. They define the *integrability index* of $\varphi$ at $a$ by

$$c(\varphi; a) = \sup\{c \geq 0 : e^{-2c\varphi} \text{ is Lebesgue integrable in a neighborhood of } a\}.$$

A classical theorem of Skoda [Sko72] states that if $\nu(\varphi; a) < 2$, then $e^{-\varphi}$ is integrable in a neighborhood of $a$ with respect to the Lebesgue measure. In terms of the quantities defined, this translates as $c(\varphi; a) \geq \nu(\varphi; a)^{-1}$. In the other direction, we also have $c(\varphi; a) \leq n\nu(\varphi; a)^{-1}$.

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