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SL_2-Action on Hilbert Schemes and Calogero–Moser Spaces

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Abstract. We study the natural GL_2-action on the Hilbert scheme of points in the plane, resp. SL_2-action on the Calogero–Moser space. We describe the closure of the GL_2-orbit, resp. SL_2-orbit, of each point fixed by the corresponding diagonal torus. We also find the character of the representation of the group GL_2 in the fiber of the Procesi bundle and its Calogero–Moser analogue over the SL_2-fixed point.

1. Introduction

The natural action of the group GL_2 on $\mathbb{C}^2$ induces a GL_2-action on $\mathrm{Hilb}^n\,\mathbb{C}^2$, the Hilbert scheme of n points in the plane. There is also a similar action of the group SL_2 on $\mathsf{X}_{\mathbf{c}}$, the Calogero–Moser space. The fixed points of the corresponding maximal torus $\mathbb{C}^* \times \mathbb{C}^*$, resp. $\mathbb{C}^*$, of diagonal matrices, are labeled by partitions. Let $y_\lambda \in \mathrm{Hilb}^n\,\mathbb{C}^2$, resp. $x_\lambda \in \mathsf{X}_{\mathbf{c}}$, denote the point labeled by a partition λ. It turns out that such a point is fixed by the group SL_2 if and only if $\lambda = (m, m-1, \ldots, 2, 1) =: \mathbf{m}$ is a *staircase* partition. In the Hilbert scheme case, this has been observed by Kumar and Thomsen [KT]. The case of the Calogero–Moser space can be deduced from the Hilbert scheme case using "hyper-Kähler rotation". A different, purely algebraic proof is given in Section 3.

The theory of rational Cherednik algebras gives an $SL_2 \times \mathfrak{S}_n$-equivariant vector bundle $\mathcal{R}$ of rank $n!$ on the Calogero–Moser space. Thus, $\mathcal{R}|_{x_{\mathbf{m}}}$, the fiber of $\mathcal{R}$ over the SL_2-fixed point, acquires the structure of a $SL_2 \times \mathfrak{S}_n$-representation. We find the character formula of this representation in terms of Kostka–Macdonald polynomials. The vector bundle $\mathcal{R}$ is an analogue of the Procesi bundle $\mathcal{P}$, a $GL_2 \times \mathfrak{S}_n$-equivariant vector bundle of rank $n!$ on $\mathrm{Hilb}^n\,\mathbb{C}^2$. Our formula agrees with the character of the representation of $GL_2 \times \mathfrak{S}_n$ in $\mathcal{P}|_{y_{\mathbf{m}}}$, the fiber of $\mathcal{P}$ over the GL_2-fixed point, obtained by Haiman [H]. It is, in fact, possible to derive our character formula for $\mathcal{R}|_{x_{\mathbf{m}}}$ from the one for $\mathcal{P}|_{y_{\mathbf{m}}}$. However, the character formula for $\mathcal{P}|_{y_{\mathbf{m}}}$, as well as the construction of the Procesi bundle itself, involves the $n!$-theorem.

In Section 2, we review some general results about SL_2-actions. In Section 3, we apply these results to show that, for any λ, the SL_2-orbit of x_λ is closed in $\mathsf{X}_{\mathbf{c}}$. The GL_2-orbit of y_λ is not closed in $\mathrm{Hilb}^n\,\mathbb{C}^2$, in general, and we describe the closure in Section 4.

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