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On the Moduli of Isotropic and Helical Minimal Immersions between Spheres

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ABSTRACT. DoCarmo–Wallach theory and its subsequent refinements assert the rich abundance of spherical minimal immersions, minimal immersions of round spheres into round spheres. A spherical minimal immersion is a conformal minimal immersion $f : S^m \to S^n$; its components are spherical harmonics of a common order p on S^m, and the conformality constant is λ_p/m, where λ_p is the pth eigenvalue of the Laplace operator on S^m. In this paper, we impose the additional constraint of "isotropy" expressed in terms of the higher fundamental forms of such immersions and determine the dimension of the respective moduli space. By the work of Tsukada, isotropy can be characterized geometrically by "helicality", constancy of initial sequences of curvatures of the image curves of geodesics under the respective spherical minimal immersions.

We first give a simple criterion for (the lowest order) isotropy of a spherical minimal immersion in terms of orthogonality relations in the third (ordinary) derivative of the image curves (Theorem A). This is then applied in the main result of this paper (Theorem B), which gives a full characterization of isotropic $SU(2)$-equivariant spherical minimal immersions of S^3 into the unit sphere of real and complex $SU(2)$-modules. Specific examples include the polyhedral minimal immersions of which the icosahedral minimal immersion (into S^{12}) is isotropic whereas its tetrahedral and octahedral cousins are not.

1. Introduction

Minimal isometric immersions of round spheres into round spheres form a rich and subtle class of objects in differential geometry studied by many authors; see [2; 4; 5; 6; 7; 8; 10; 12; 15; 16; 18; 17; 19; 20; 25; 24; 26; 27] and, for a more complete list, the bibliography at the end of the second author's monograph [23]. Such immersions can be written as $f : S^m_\kappa \to S_V$ of the round m-sphere S^m_κ of (constant) curvature $\kappa > 0$ into the unit sphere S_V of a Euclidean vector space V or, scaling the domain sphere S^m_κ to radius one, as minimal immersions $f : S^m \to S_V$ with *homothety constant* $1/\kappa$. By minimality, the components $\alpha \circ f$, $\alpha \in V^*$ (the

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