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Weak Amenability of the Central Beurling Algebras on $[\mathrm{FC}]^-$ Groups

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Abstract. We study weak amenability of central Beurling algebras $ZL^1(G,\omega)$. The investigation is a natural extension of the known work on the commutative Beurling algebra $L^1(G,\omega)$. For $[\mathrm{FC}]^-$ groups, we establish a necessary condition, and for $[\mathrm{FD}]^-$ groups, we give sufficient conditions for the weak amenability of $ZL^1(G,\omega)$. For a compactly generated $[\mathrm{FC}]^-$ group with polynomial weight $\omega_\alpha(x) = (1+|x|)^\alpha$, we prove that $ZL^1(G,\omega_\alpha)$ is weakly amenable if and only if $\alpha < 1/2$.

1. Introduction

Let G be a locally compact group. As it is customary, two functions equal to each other almost everywhere on G with respect to the Haar measure will be regarded as the same. We denote the integral of a function f on a (Borel-)measurable subset K of G against a fixed left Haar measure by $\int_K f\,dx$. The space of all complex-valued Haar-integrable functions on G is denoted by $L^1(G)$. A weight on G is a Borel-measurable function $\omega : G \to \mathbb{R}^+$ satisfying

$$\omega(xy) \le \omega(x)\omega(y) \quad (x, y \in G).$$

Given a weight ω on G, we consider the space $L^1(G,\omega)$ of all complex-valued Haar-measurable functions f on G that satisfy

$$\|f\|_\omega = \int |f(x)|\omega(x)\,dx < \infty.$$

With the convolution product $*$ and the norm $\|\cdot\|_\omega$, $L^1(G,\omega)$ is a Banach algebra, called a Beurling algebra on G. When $\omega = 1$, this is simply the group algebra $L^1(G)$. Let $ZL^1(G,\omega)$ be the closed subalgebra of $L^1(G,\omega)$ consisting of all $f \in L^1(G,\omega)$ such that $f^g = f$ for all $g \in G$, where $f^g(x) = f(g^{-1}xg)$ $(x \in G)$. Then $ZL^1(G,\omega)$ is a commutative Banach algebra, called a *central Beurling algebra* [19]. Indeed, $ZL^1(G,\omega)$ is the center of $L^1(G,\omega)$. It is well known that $ZL^1(G,\omega)$ is nontrivial if and only if G is an [IN] group [22].

From [8, Rem. 8.8], a measurable weight ω on G is always equivalent to a continuous weight $\tilde{\omega}$ on G, where the equivalence means that there are constants $c_1, c_2 > 0$ such that $c_1\omega(x) \le \tilde{\omega}(x) \le c_2\omega(x)$ for almost all $x \in G$. The equivalence implies that the respective Beurling algebras $L^1(G,\omega)$ and $L^1(G,\tilde{\omega})$ are

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